# ROUGH VOLATILITY, PATH-DEPENDENT PDES AND WEAK RATES OF CONVERGENCE 

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#### Abstract

In the setting of stochastic Volterra equations, and in particular rough volatility models, we show that conditional expectations are the unique classical solutions to path-dependent PDEs. The latter arise from the functional Itô formula developed by [Viens, F., \& Zhang, J. (2019). A martingale approach for fractional Brownian motions and related path dependent PDEs. Ann. Appl. Probab.]. We then leverage these tools to study weak rates of convergence for discretised stochastic integrals of smooth functions of a Riemann-Liouville fractional Brownian motion with Hurst parameter $H \in$ $(0,1 / 2)$. These integrals approximate log-stock prices in rough volatility models. We obtain weak error rates of order 1 if the test function is quadratic and of order $H+1 / 2$ for smooth test functions.


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## 1. Introduction

1.1. Motivation. Until and unless softwares become capable of handling infinite quantities in finite time, numerical error analysis justifies the application of continuous-time models to discretised realworld applications. In this paper, we consider a time-grid of $(N+1)$ equally spaced points $\left(t_{i}:=\frac{i T}{N}\right)_{i=0}^{N}$ and we study the convergence, as $\Delta_{t}=\frac{T}{N}$ goes to zero, of the Euler approximation

$$
\begin{equation*}
\bar{X}_{t_{0}}=x_{0}, \quad \bar{X}_{t_{i+1}}=\bar{X}_{t_{i}}+\psi\left(V_{t_{i}}\right)\left(B_{t_{i+1}}-B_{t_{i}}\right)-\frac{1}{2} \psi\left(V_{t_{i}}\right)^{2} \Delta_{t} \tag{1.1}
\end{equation*}
$$

to the original rough volatility model, where $X$ represents the log-price,

$$
\begin{equation*}
X_{t}=x_{0}+\int_{0}^{t} \psi\left(V_{r}\right) \mathrm{d} B_{r}-\frac{1}{2} \int_{0}^{t} \psi\left(V_{r}\right)^{2} \mathrm{~d} r, \quad V_{t}=\int_{0}^{t}(t-r)^{H-\frac{1}{2}} \mathrm{~d} W_{r} \tag{1.2}
\end{equation*}
$$

Here, $x_{0} \in \mathbb{R}, B$ and $W$ are correlated Brownian motions with the natural filtration $\left(\mathcal{F}_{t}\right)_{t \in[0, T]}$, and $V$ is a Gaussian (Volterra) process with known covariance function, which can therefore be exactly sampled at discrete-time points by Cholesky decomposition. Anyhow, the pivotal parameter is $H$, which controls the Hölder regularity of $V$. When $H=\frac{1}{2}$, one recovers the well-known Markov and semimartingale theories. The singular case $H \in\left(0, \frac{1}{2}\right)$, supported by empirical data under both historical and pricing measures 4. 9, 14, 19, 20, 32, 37, is precisely where both theoretical and numerical analyses go haywire. While $X$ remains a semimartingale, $V$ fails to be so, and an application of Itô's formula with $\psi$ Lipschitz continuous shows that

$$
\mathbb{E}\left[\left|X_{T}-\bar{X}_{T}\right|^{2}\right] \lesssim \sum_{i=0}^{N-1} \int_{t_{i}}^{t_{i+1}} \mathbb{E}\left[\left|V_{r}-V_{t_{i}}\right|^{2}\right] \mathrm{d} r \lesssim \Delta_{t}^{2 H}
$$

namely that the strong rate of convergence is of order $H$. When $H$ is close to zero, as is generally agreed upon in the community, a practical implementation would not converge in a reasonable amount of time (dividing the error by 2 implies multiplying $N$ by $2^{1 / H}$ ). This shortcoming is confirmed by the lower bound of [26, and in the more general framework of Stochastic Volterra Equations (SVEs) by 22. 31 (rate $H$ and $2 H$ for Euler and Milstein schemes respectively) and 3) (rate $4 H / 3$ with a multifactor approximation).
1.2. Weak rates. Fortunately, the financial applications we have in mind, such as option pricing, only require to approximate quantities of the form $\mathbb{E}\left[\phi\left(X_{T}\right)\right]$, for a payoff function $\phi$. This falls into the realm of the weak rate of convergence which is usually much higher than its strong counterpart. Most friendly stochastic integrals, e.g. Itô's SDEs where $H=\frac{1}{2}$, exhibit 1 for the former and $1 / 2$ for the latter. One then naturally wonders how the weak error rate evolves in the inhospitable interval $H \in\left(0, \frac{1}{2}\right)$. In particular, a positive lower bound is needed to justify the use of simulation schemes and thereby to mitigate the high cost induced by such a tiny additional parameter.

Unfortunately, the analysis of the weak rate turns out to be an intricate problem, even in relatively simple settings. For $H>\frac{1}{2}$, 8 obtained a weak rate of order $H$ where the volatility follows a fractional Ornstein-Uhlenbeck process. Regarding rough volatility models, the rough Donsker theorem of 21, with weak rate $H$, was designed in an age when bare convergence was already an achievement, while the quantization methods developed in [10 only applies to VIX derivatives. Regarding Euler schemes, the following table is an exhaustive literature review of the results obtained so far.

| Authors | Weak rate | Assumptions |
| :--- | :---: | :--- |
| Bayer, Hall, Tempone 6 | $H+\frac{1}{2}$ | Linear variance $(\psi(x)=x)$ and bounded payoff $\left(\phi \in \mathcal{C}_{b}^{\left\lceil\frac{1}{H}\right\rceil}\right)$ |
| Bayer, Fukasawa, Nakahara [5 | $H+\frac{1}{2}$ | Linear variance $(\psi(x)=x)$ and regular payoff $\left(\phi \in \mathcal{C}_{p}^{2+\left\lceil\frac{1}{2 H}\right\rceil}\right)$ |
| Gassiat [18] | $3 H+\frac{1}{2}$ | Linear variance $(\psi(x)=x)$ and bounded payoff $\left(\phi \in \mathcal{C}_{b}^{3+2\left\lceil\frac{1}{4 H}\right\rceil}\right)$ <br> regular variance $\left(\psi \in \mathcal{C}_{b}^{2}\right)$ and cubic payoff $\left(\phi(x)=x^{3}\right)$ |
| Friz, Salkeld, Wagenhofer [16] | $3 H+\frac{1}{2}$ | Regular variance $\left(\psi \in \mathcal{C}^{N}, N \in \mathbb{N}\right)$ and polynomial payoff $(\phi(x)=$ <br> $\left.x^{n}, n \leq N\right)$ |

They give a positive answer to the question of the lower bound: a minimum of $1 / 2$ is achieved in all these papers, even supplemented by an unexpected $3 H$ in the last two. On the other hand, all of them rely one way or another on the given structure of the model (the last column) to derive explicit computations. Friz, Salkeld and Wagenhofer 16 built on previous ideas by Gassiat 18, which is itself in a similar spirit as the duality approach in [11, to write down an explicit formula for the moments of the stochastic integral in (1.2). In both cases, a precise fractional analysis is necessary to wind up with a sweet $1 / 2+3 H$ weak rate.

This problem is difficult because, here more than elsewhere, standard techniques for diffusions rely on Itô's formula or on the Markov property, and in particular on PDE methods 33, 34, 28. Inspired by the functional Itô formula developed in 35 , and the resulting path-dependent PDEs (PPDEs) studied in [36, we decide to explore a PDE approach analogous to the Markov case. The aim of this paper is then twofold:
(1) We present the general PPDE theory for SVEs (Section 2.3) and prove that it applies to rough volatility models (Section 2.4).
(2) For the Euler approximation (1.1) with no drift, we derive weak error rates of order higher than $1 / 2+H$, assuming that $\phi, \psi \in C^{\infty}$ with polynomial and exponential growth respectively (Theorem 3.1).

Relaxing the requirements on $\phi$ and $\psi$ is the main achievement of this work compared to former results as it breaks away from the polynomial setting. Although the error analysis is restricted to the case without drift and exactly sampled variance for conciseness, the PDE approach should stretch beyond this setting without difficulty, and even has the potential to extend to larger classes of SVEs.
1.3. Path-dependent PDEs. Viens and Zhang 35 were interested in understanding the pathdependent structure of conditional expectations of functionals of Volterra processes. Let us start by explaining the idea at the core of their paper in a simple setting. For $0 \leq t \leq s \leq T$, the authors discarded the decomposition $V_{s}=V_{t}+\left[V_{s}-V_{t}\right]$ as $V$ does not have a flow or the Markov property. Instead, they promoted the decomposition

$$
V_{s}=\int_{0}^{s} K(s, r) \mathrm{d} W_{r}=\underbrace{\int_{0}^{t} K(s, r) \mathrm{d} W_{r}}_{=: \Theta_{s}^{t}}+\underbrace{\int_{t}^{s} K(s, r) \mathrm{d} W_{r}}_{=: I_{s}^{t}},
$$

which is an orthogonal decomposition in the sense that, for all $t \leq s, \Theta_{s}^{t}$ is $\mathcal{F}_{t}$-measurable and $I_{s}^{t}$ is independent of $\mathcal{F}_{t}$. In financial terms, $\Theta_{s}^{t}=\mathbb{E}\left[V_{s} \mid \mathcal{F}_{t}\right]$ corresponds to the forward variance. Moreover, it turns out that $\Theta^{t}$ encodes precisely the path-dependence one needs to express the conditional expectation. Consider (1.2) and ignore the drift for clarity, then

$$
\begin{aligned}
\mathbb{E}\left[\phi\left(X_{T}\right) \mid \mathcal{F}_{t}\right] & =\mathbb{E}\left[\phi\left(X_{t}+\int_{t}^{T} \psi\left(V_{s}\right) \mathrm{d} B_{s}\right) \mid \mathcal{F}_{t}\right] \\
& =\mathbb{E}\left[\phi\left(X_{t}+\int_{t}^{T} \psi\left(\Theta_{s}^{t}+I_{s}^{t}\right) \mathrm{d} B_{s}\right) \mid X_{t}, \Theta_{[t, T]}^{t}\right] \\
& =u\left(t, X_{t}, \Theta_{[t, T]}^{t}\right)
\end{aligned}
$$

where $u:[0, T] \times \mathbb{R} \times C([t, T]) \rightarrow \mathbb{R}$ is defined as

$$
\begin{equation*}
u(t, x, \omega):=\mathbb{E}\left[\phi\left(X_{T}\right) \mid X_{t}=x, \Theta^{t}=\omega\right] \tag{1.3}
\end{equation*}
$$

This means that the conditional expectation is only a function of $X_{t}$ and $\Theta_{[t, T]}^{t}$, in other words we do not need to take into account the past of $X$ and $V$, respectively $X_{[0, t)}$ and $V_{[0, t)}$. We pay the recovery of the Markov property of $\left(X, \Theta^{t}\right)$ (see [17] for a proof) by lifting the state space to a path space. Let us mention that such a representation was achieved the same year by [15] for the rough Heston model using different techniques and that, in the affine framework, this also induces a stochastic PDE representation 112 .

In a more general setting where $\boldsymbol{X}$ solves an SVE like 2.1 and $\boldsymbol{\Theta}^{t}$ is the appropriate $\mathcal{F}_{t}$-measurable projection, the main result of [35] is a functional Itô formula for functions of the concatenated process $\boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}:=\boldsymbol{X} \mathbb{1}_{[0, t]}+\boldsymbol{\Theta}^{t} \mathbb{1}_{[t, T]}$ taking values in $C\left([0, T], \mathbb{R}^{d}\right)$, see Section 2 . This involves Fréchet derivatives where the path is only perturbed on $[t, T]$ in the direction of the kernel. Using BSDE techniques and an intermediary process, 36] demonstrated that conditional expectations of the latter are classical solutions to (semilinear parabolic) path-dependent PDEs, as alluded to in [35. As we detail in Remark 2.18, the Fréchet derivative generalises Dupire's vertical derivative [13] as the path is not frozen and the direction not constant. The Volterra case leads to a different class of PPDEs which do not lie in the scope of previous existence results 30, 29.

We adapt the proof of 36 to include singular kernels, derive uniqueness of the solution, and show the connection with the original SVE. This confirms that the representation (1.3) holds for a broad class of SVEs. The caveat lies in the stringent assumptions made on the road to Itô's formula: the function $u$ needs to be twice Fréchet differentiable with particular regularity conditions. Exploiting the smoothness of the payoff $\phi$ and of the volatility function $\psi$, we are however able to verify these for rough volatility models of interest $(1.2)$, and thus prove well-posedness of the pricing PPDE $(2.23)$. Outside weak error rates, we believe this opens the gates to a number of PDE applications for rough volatility, such as but not limited to, stochastic control and optimisation, numerical methods 24, hedging or integration by parts formula.
1.4. Method of proof. The basic idea of our proof consists in expressing the error $\mathbb{E}\left[\phi\left(X_{T}\right)\right]-$ $\mathbb{E}\left[\phi\left(\bar{X}_{T}\right)\right]$ as a telescopic sum of conditional expectations between successive discretisation points, as in [28, Section 7.6] and [6] for instance. We can apply the functional Itô formula to the Euler approximation between $t_{i}$ and $t_{i+1}$ and then cancel the time derivative of $u$ thanks to the PPDE; this is presented in Proposition 3.4. The problem boils down to studying local errors of the type

$$
\begin{equation*}
\mathbb{E}\left[\left(\psi^{2}\left(V_{t}\right)-\psi^{2}\left(V_{t_{i}}\right)\right) \phi^{\prime \prime}\left(X_{T}\right)\right] \tag{1.4}
\end{equation*}
$$

Only a sharp and meticulous analysis can exploit this difference without exhibiting the strong rate. Indeed, the variance process is not a semimartingale so we cannot apply Itô's formula again. Furthermore, other approximations such as Taylor's formula lead to the appearance of singular kernels in the wrong places, so we could not get a rate better than $2 H$. On the other hand, even though $\mathbb{E}\left[\psi^{2}\left(V_{t}\right)-\psi^{2}\left(V_{t_{i}}\right)\right]$ only yields a local weak rate of $2 H$, the passage to the global rate (after integrating over $t$ and summing
over $i$ ) achieves rate one (Lemma 3.7). This trick allows us to recover this optimal rate for quadratic payoffs, and inspires us to look for ways of disentangling (1.4).

We decide to perform a joint chaos expansion (3.5) that achieves this goal but becomes quite intricate as it generates Malliavin derivatives of all orders (and thus requires $\phi$ and $\psi$ to be smooth). The remainder of the proof consists in a careful analysis of the terms of the expansion and a precise estimation of their bounds (Proposition 3.10). It involves a mixture of Malliavin calculus, combinatorics and a subtle analysis and integration of the multiple kernels. Once the dust settles, the weak rate of $1 / 2+H$ essentially arises from integrals of the form $\int_{0}^{t}\left|K(t, r)-K\left(t_{i}, r\right)\right| \mathrm{d} r$ and the regularity of $\mathbb{E}\left[\psi\left(V_{t}\right)\right]$ summarised in the hypothesis (3.3).

Through the course of our computations, we notice that multiple integrals of kernels-coming from the Malliavin derivative of the volatility - appear both in our paper and in [16]. The estimates presented in Section 5.2 are reminiscent of Equations (4.2) and (4.3) in that paper.

In a nutshell, this paper is a tale that starts from a numerical analysis problem, then dives into stochastic analysis before bouncing back on PDE theory, goes on to explore a region of Malliavin calculus, adds a salty touch of combinatorics and ends its course with fractional calculus.
1.5. Outlook. In view of the recent results [18, 16, we acknowledge that our rate is not optimal but it seemed too hard to reach $1 / 2+3 H$ with the chaos expansion approach. Hopefully an alternative way of studying the difference (1.4) may achieve that goal. We work in the simplified setting with no drift and exactly sampled variance but we expect that the additional finite variation term or discretisation of the Gaussian process (e.g. hybrid scheme [7] as in [18) should lead to similar results. In principle, the same route can also be employed to study the weak error of more general SVEs; it is however a wide open question whether that rate should consistently be higher than $1 / 2$.
1.6. Organisation of the paper. The rest of the paper is structured as follows. In Section 2.2, we introduce the framework and definitions needed to extend the Itô formula to functionals with superpolynomial growth. In Section 2.3, the connection with the path-dependent PDE is established, as in the Markovian setting. We apply these results to rough volatility models in Section 2.4 and postpone the proofs to Section 4. In Section 3 we state our main result for the weak rates of convergence, whose proof is carried out in details in Sections 56 and 7. Finally, technical lemmas and important constants are gathered in the appendix.

## 2. The functional Itô formula and the path-Dependent PDE

2.1. Notations. We write, for $m, n \in \mathbb{N}$ with $m<n, \llbracket m, n \rrbracket:=\{m, \ldots, n\}$ and $\llbracket n \rrbracket:=\llbracket 1, n \rrbracket=$ $\{1, \ldots, n\}$. We fix a finite time horizon $T>0$ and denote the corresponding time interval as $\mathbb{T}:=[0, T]$. For a couple of topological spaces $\mathcal{Y}$ and $\mathcal{Z}$, the set $C^{0}(\mathcal{Y}, \mathcal{Z})\left(\right.$ resp. $\left.D^{0}(\mathcal{Y}, \mathcal{Z})\right)$ represents the set of continuous (resp. cadlag) functions from $\mathcal{Y}$ to $\mathcal{Z}$. We write $f(x) \lesssim g(x)$ for two positive functions $f, g$ if there exists $c>0$ independent of $x$ such that $f(x) \leq c g(x)$. We only use this notation when it is clear that the constant $c$ is inconsequential. We define $\mathfrak{b}_{p}$ to be the BDG constant for any $p>1$.

We consider multi-dimensional stochastic Volterra equations (SVE) given by

$$
\begin{equation*}
\boldsymbol{X}_{t}=x+\int_{0}^{t} b\left(t, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\int_{0}^{t} \sigma\left(t, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r} \tag{2.1}
\end{equation*}
$$

taking values in $\mathbb{R}^{d}$, with $d \geq 1$. The coefficients $b: \mathbb{T}^{2} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and $\sigma: \mathbb{T}^{2} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times m}$ are Borel-measurable functions, and $\boldsymbol{W}$ is an $m$-dimensional Brownian motion on the filtered probability space $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \in \mathbb{T}}, \mathbb{P}\right)$ satisfying the usual conditions. For each $t \in \mathbb{T}$, we introduce the key $\mathcal{F}_{t^{-}}$ measurable process $\left(\boldsymbol{\Theta}_{s}^{t}\right)_{s \geq t}$ :

$$
\begin{equation*}
\boldsymbol{\Theta}_{s}^{t}=x+\int_{0}^{t} b\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\int_{0}^{t} \sigma\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r} . \tag{2.2}
\end{equation*}
$$

We highlight the following important property: Fixing $s \in \mathbb{T}$ and viewing the index $t \in[0, s]$ as time, $\left(\boldsymbol{\Theta}_{s}^{t}-\int_{0}^{t} b\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r\right)_{t \in[0, s]}$ is a martingale (provided the right integrability conditions).
2.2. Definitions and the functional Itô formula. We need to introduce a number of notations that lead to the functional Itô formula. The following is a summary of 35. Section 3.1] with the small twist that we allow for faster than polynomial growth. We define the following spaces:

$$
\begin{array}{ll}
\mathcal{W}:=C^{0}\left(\mathbb{T}, \mathbb{R}^{d}\right), & \overline{\mathcal{W}}:=\mathrm{D}^{0}\left(\mathbb{T}, \mathbb{R}^{d}\right), \quad \mathcal{W}_{t}=C^{0}\left([t, T], \mathbb{R}^{d}\right) \\
\Lambda:=\mathbb{T} \times \mathcal{W}, & \bar{\Lambda}:=\left\{(t, \boldsymbol{\omega}) \in \mathbb{T} \times \overline{\mathcal{W}}:\left.\boldsymbol{\omega}\right|_{[t, T]} \in \mathcal{W}_{t}\right\} \\
\|\boldsymbol{\omega}\|_{\mathbb{T}}=\sup _{t \in \mathbb{T}}\left|\boldsymbol{\omega}_{t}\right|, & \boldsymbol{d}\left((t, \boldsymbol{\omega}),\left(t^{\prime}, \boldsymbol{\omega}^{\prime}\right)\right):=\left|t-t^{\prime}\right|+\left\|\boldsymbol{\omega}-\boldsymbol{\omega}^{\prime}\right\|_{\mathbb{T}}
\end{array}
$$

For two paths $\boldsymbol{\omega}, \boldsymbol{\theta}$ on $\mathbb{T}$ we define their concatenation at time $t \in \mathbb{T}$ as

$$
\boldsymbol{\omega} \otimes_{t} \boldsymbol{\theta}:=\boldsymbol{\omega} \mathbb{1}_{[0, t]}+\boldsymbol{\theta} \mathbb{1}_{[t, T]} .
$$

Let $C^{0}(\bar{\Lambda}):=C^{0}(\bar{\Lambda}, \mathbb{R})$ denote the set of functions $u: \bar{\Lambda} \rightarrow \mathbb{R}$ continuous under $\boldsymbol{d}$ and define the right time derivative

$$
\partial_{t} u(t, \boldsymbol{\omega}):=\lim _{\varepsilon \downarrow 0} \frac{u(t+\varepsilon, \boldsymbol{\omega})-u(t, \boldsymbol{\omega})}{\varepsilon}
$$

for all $(t, \boldsymbol{\omega}) \in \bar{\Lambda}$, provided the limit exists. We also recall the spatial derivative $\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega})$ with respect to $\boldsymbol{\omega}$, in the direction of $\eta \in \mathcal{W}_{t}$ :

$$
\begin{equation*}
\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \eta\right\rangle:=\lim _{\varepsilon \downarrow 0} \frac{u\left(t, \boldsymbol{\omega}+\varepsilon \eta \mathbb{1}_{[t, T]}\right)-u(t, \boldsymbol{\omega})}{\varepsilon} . \tag{2.3}
\end{equation*}
$$

This is both a Fréchet and a Gateaux derivative with respect to $\boldsymbol{\omega} \mathbb{1}_{[t, T]}$. We similarly define the second derivative $\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\eta^{(1)}, \eta^{(2)}\right)\right\rangle$.

The following assumption ensures well-posedness of (2.1):
Assumption 2.1. The coefficients $b, \sigma$ are such that:
(i) The SVE (2.1) has a weak solution $(\boldsymbol{X}, \boldsymbol{W})$;
(ii) For all $p \geq 1, \mathbb{E}\left[\|\boldsymbol{X}\|_{\mathbb{T}}^{p}\right]=\mathbb{E}\left[\sup _{t \in \mathbb{T}}\left|\boldsymbol{X}_{t}\right|^{p}\right]<\infty$.

This assumption, taken from 35, only requires $\boldsymbol{X}$ to have finite moments which forces to take into account only functions of $\boldsymbol{X}$ with polynomial growth. We would like to relax this assumption such that super-polynomial growth is allowed as long as the function is still integrable (a Gaussian process has exponential moments for instance). For this reason, we introduce the following function space.
Definition 2.2. A function $G: \mathbb{R}^{d} \rightarrow \mathbb{R}$ belongs to $\mathcal{X}$ if $\mathbb{E}\left[\|G(\boldsymbol{X})\|_{\mathbb{T}}^{p}\right]<\infty$, for all $p \geq 1$.
Example 2.3. If the coefficients $b, \sigma$ have linear growth in space then standard arguments show that $\mathbb{E}\left[\left\|\boldsymbol{X}_{t}\right\|_{\mathbb{T}}^{p}\right]$ is finite for all $p>1$, so that the set $\mathcal{X}$ corresponding to these specifications includes polynomials.

A thorough examination of the proofs in 35 confirms that their functional Itô formula still stand with the new growth definitions that follow.
Definition 2.4. Let $u \in C^{0}(\bar{\Lambda})$ such that $\partial_{\boldsymbol{\omega}} u$ exists for all $(t, \boldsymbol{\omega}) \in \bar{\Lambda}$. For $G \in \mathcal{X}$, we say that $\partial_{\boldsymbol{\omega}} u$ has $G$-growth if

$$
\left|\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \eta\right\rangle\right| \lesssim\|G(\boldsymbol{\omega})\|_{\mathbb{T}}\left\|\eta \mathbb{1}_{[t, T]}\right\|_{\mathbb{T}}, \quad \text { for all } \eta \in \mathcal{W}, \text { for all } t \in \mathbb{T}
$$

and similarly for $\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega})$ :
$\left|\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\eta^{(1)}, \eta^{(2)}\right)\right\rangle\right| \lesssim\|G(\boldsymbol{\omega})\|_{\mathbb{T}}\left\|\eta^{(1)} \mathbb{1}_{[t, T]}\right\|_{\mathbb{T}}\left\|\eta^{(2)} \mathbb{1}_{[t, T]}\right\|_{\mathbb{T}}, \quad$ for all $\eta^{(1)}, \eta^{(2)} \in \mathcal{W}$, for all $t \in \mathbb{T}$.
Definition 2.5. We say $u \in C^{1,2}(\bar{\Lambda}) \subset C^{0}(\bar{\Lambda})$ if $\partial_{t} u, \partial_{\boldsymbol{\omega}} u, \partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u$ exist and are continuous on $\bar{\Lambda}$. Moreover, $u \in C_{+}^{1,2}(\bar{\Lambda}) \subset C^{1,2}(\bar{\Lambda})$ if there exists $G \in \mathcal{X}$ such that all derivatives of $u$ have $G$-growth.
Assumption 2.6. For $\varphi=b, \sigma$, for every $t \in(s, T]$, with $s \in \mathbb{T}, \partial_{t} \varphi(t ; s, \cdot)$ exists and there exist $G \in \mathcal{X}$ and $H \in\left(0, \frac{1}{2}\right)$ such that, for all $x \in \mathbb{R}^{d}$,

$$
|\varphi(t ; s, x)| \lesssim G(x)(t-s)^{H-\frac{1}{2}}, \quad\left|\partial_{t} \varphi(t ; s, x)\right| \lesssim G(x)(t-s)^{H-\frac{3}{2}}
$$

Example 2.7. The model we have in mind for relaxing the polynomial growth assumption of [35] is a rough volatility model where the second component is a Gaussian (Volterra) process and the first (the log-price) is a non-Gaussian stochastic integral. The latter has finite moments while the former has finite exponential moments (both in supremum norm), hence the set $\mathcal{X}$ corresponding to this framework includes functions with polynomial growth in the first component and exponential growth in the second, such as

$$
G\left(\boldsymbol{\omega}_{t}\right)=C\left(1+\left|\boldsymbol{\omega}_{t}^{1}\right|+\mathrm{e}^{\boldsymbol{\omega}_{t}^{2}}\right), \quad \text { for some } C>0
$$

Definition 2.8. We say that $u \in C_{+, \alpha}^{1,2}(\Lambda)$, with $\alpha \in(0,1)$, if there exists a continuous extension of $u \in C_{+}^{1,2}(\bar{\Lambda})$, a growth function $G \in \mathcal{X}$ and a modulus of continuity function $\varrho \leq G$ such that: for any $0 \leq t<T, 0<\delta \leq T-t$, and $\eta, \eta_{1}, \eta_{2} \in \mathcal{W}_{t}$ with supports contained in $[t, t+\delta]$,
(i) for any $\boldsymbol{\omega} \in \overline{\mathcal{W}}$ such that $\boldsymbol{\omega} \mathbb{1}_{[t, T]} \in \mathcal{W}_{t}$,

$$
\begin{aligned}
\left|\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \eta\right\rangle\right| & \leq\|G(\boldsymbol{\omega})\|_{\mathbb{T}}\|\eta\|_{\mathbb{T}} \delta^{\alpha} \\
\left|\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\eta_{1}, \eta_{2}\right)\right\rangle\right| & \leq\|G(\boldsymbol{\omega})\|_{\mathbb{T}}\left\|\eta_{1}\right\|_{\mathbb{T}}\left\|\eta_{2}\right\|_{\mathbb{T}} \delta^{2 \alpha}
\end{aligned}
$$

(ii) for any other $\boldsymbol{\omega}^{\prime} \in \overline{\mathcal{W}}$ such that $\boldsymbol{\omega}^{\prime} \mathbb{1}_{[t, T]} \in \mathcal{W}_{t}$,

$$
\begin{aligned}
\left|\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega})-\partial_{\boldsymbol{\omega}} u\left(t, \boldsymbol{\omega}^{\prime}\right), \eta\right\rangle\right| & \leq\left(\|G(\boldsymbol{\omega})\|_{\mathbb{T}}+\left\|G\left(\boldsymbol{\omega}^{\prime}\right)\right\|_{\mathbb{T}}\right)\|\eta\|_{\mathbb{T}} \varrho\left(\left\|\boldsymbol{\omega}-\boldsymbol{\omega}^{\prime}\right\|_{\mathbb{T}}\right) \delta^{\alpha} \\
\left|\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega})-\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u\left(t, \boldsymbol{\omega}^{\prime}\right),\left(\eta_{1}, \eta_{2}\right)\right\rangle\right| & \leq\left(\|G(\boldsymbol{\omega})\|_{\mathbb{T}}+\left\|G\left(\boldsymbol{\omega}^{\prime}\right)\right\|_{\mathbb{T}}\right)\left\|\eta_{1}\right\|_{\mathbb{T}}\left\|\eta_{2}\right\|_{\mathbb{T}} \varrho\left(\left\|\boldsymbol{\omega}-\boldsymbol{\omega}^{\prime}\right\|_{\mathbb{T}}\right) \delta^{2 \alpha}
\end{aligned}
$$

(iii) For any $\boldsymbol{\omega} \in \overline{\mathcal{W}},\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \eta\right\rangle$ and $\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\eta_{1}, \eta_{2}\right)\right\rangle$ are continuous in $t$.

Remark 2.9. In 35, Definition 3.4], the modulus of continuity is assumed bounded. This property can be replaced by a condition on its growth as we did above.

Remark 2.10. The time continuity is not necessary to apply Itô's formula but we need it in the proof of Proposition 2.14 .

We intend to deal with singular coefficients that shall satisfy Assumption 2.6 but may not be continuous on the diagonal. Hence, for $\varphi=b, \sigma$ and $\delta>0$, we introduce the truncated functions

$$
\varphi^{\delta}(s ; t, x):=\varphi(s \wedge(t+\delta) ; t, x)
$$

and for $u \in C_{+, \alpha}^{1,2}$ the spatial derivatives

$$
\begin{align*}
\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \varphi^{t, \boldsymbol{\omega}}\right\rangle & :=\lim _{\delta \downarrow 0}\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \varphi^{\delta, t, \boldsymbol{\omega}}\right\rangle, \quad \varphi=b, \sigma,  \tag{2.4}\\
\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\sigma^{t, \boldsymbol{\omega}} \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle & :=\lim _{\delta \downarrow 0}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\sigma^{\delta, t, \boldsymbol{\omega}} \sigma^{\delta, t, \boldsymbol{\omega}}\right)\right\rangle,
\end{align*}
$$

where $\varphi^{t, \boldsymbol{\omega}}(s):=\varphi\left(s ; t, \boldsymbol{\omega}_{t}\right)$. We can finally display the functional Itô formula from 35. Theorem 3.17].
Theorem 2.11. Let Assumptions 2.1 and 2.6 hold. Assume $u \in C_{+, \alpha}^{1,2}$, with $\alpha>\frac{1}{2}-H$ and $H \in\left(0, \frac{1}{2}\right)$. Then the spatial derivatives (2.4) exist and the following functional Ito formula holds

$$
\begin{align*}
\mathrm{d} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right)= & \partial_{t} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right) \mathrm{d} t+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right),\left(\sigma^{t, \boldsymbol{X}}, \sigma^{t, \boldsymbol{X}}\right)\right\rangle \mathrm{d} t \\
& +\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right), b^{t, \boldsymbol{X}}\right\rangle \mathrm{d} t+\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right), \sigma^{t, \boldsymbol{X}}\right\rangle \mathrm{d} \boldsymbol{W}_{t} \tag{2.5}
\end{align*}
$$

Remark 2.12. One can show that if $\boldsymbol{\omega}$ is $\mathbb{R}^{d}$-valued, then for $\eta$ (living in $\mathbb{R}^{d}$ as well), we have

$$
\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \eta\right\rangle=\sum_{i=1}^{d}\left\langle\partial_{\boldsymbol{\omega}_{i}} u(t, \boldsymbol{\omega}), \eta_{i}\right\rangle \quad \text { and } \quad\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),(\eta, \eta)\right\rangle=\sum_{i, j=1}^{d}\left\langle\partial_{\boldsymbol{\omega}_{i} \boldsymbol{\omega}_{j}} u(t, \boldsymbol{\omega}),\left(\eta_{i}, \eta_{j}\right)\right\rangle .
$$

Furthermore, if $\boldsymbol{\eta}$ is matrix valued, say $\boldsymbol{\eta} \in \mathbb{R}^{d \times m}$, then

$$
\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \boldsymbol{\eta}\right\rangle=\left(\sum_{i=1}^{d}\left\langle\partial_{\boldsymbol{\omega}_{i}} u(t, \boldsymbol{\omega}), \boldsymbol{\eta}_{i, j}\right\rangle\right)_{j=1}^{m} \quad \text { and }
$$

$$
\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),(\boldsymbol{\eta}, \boldsymbol{\eta})\right\rangle=\sum_{k=1}^{m} \sum_{i, j=1}^{d}\left\langle\partial_{\boldsymbol{\omega}_{i} \boldsymbol{\omega}_{j}} u(t, \boldsymbol{\omega}),\left(\boldsymbol{\eta}_{j, k}, \boldsymbol{\eta}_{i, k}\right)\right\rangle .
$$

2.3. A path-dependent PDE. As in the Markovian case, this Itô formula leads to natural connections with PDEs. In this section, we rigorously prove that conditional expectations (equivalently, solutions to a BSDE) are solutions to path-dependent PDEs. This was predicted by [35] Section 4.2] where they also showed the reverse implication, knwon as Feynman-Kac formula. The crucial aspect is to show time differentiability, for which we follow [36. Theorem 3.2], where a similar result is presented for regular kernels. This requires to introduce new variables and notations. Define the forward-backward SVE for $\boldsymbol{\omega} \in \mathcal{W}$ and all $0 \leq t \leq s \leq T$ :

$$
\begin{align*}
\boldsymbol{X}_{s}^{t, \boldsymbol{\omega}} & =\boldsymbol{x}_{s}+\int_{t}^{s} b\left(s, r, \boldsymbol{X}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r+\int_{t}^{s} \sigma\left(s, r, \boldsymbol{X}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} \boldsymbol{W}_{r}  \tag{2.6}\\
\boldsymbol{Y}_{s}^{t, \boldsymbol{\omega}} & =\Phi\left(\boldsymbol{X}^{s, \boldsymbol{\omega}}\right)+\int_{s}^{T} f\left(r, \boldsymbol{X}^{t, \boldsymbol{\omega}}, \boldsymbol{Y}_{r}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r-\int_{s}^{T} \boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}} \mathrm{~d} \boldsymbol{W}_{r} . \tag{2.7}
\end{align*}
$$

The forward process $\boldsymbol{X}$ lives in $\mathbb{R}^{d}$ and the backward one $\boldsymbol{Y}$ in $\mathbb{R}^{d^{\prime}}$ with $d, d^{\prime} \in \mathbb{N}$. Moreover $\Phi$ maps $\mathcal{W}$ to $\mathbb{R}^{d^{\prime}}$ and $f: \mathbb{T} \times \mathcal{W} \times \mathbb{R}^{d^{\prime}} \times \mathbb{R}^{d^{\prime} \times d} \rightarrow \mathbb{R}^{d^{\prime}}$ is measurable in all variables. Under the following assumptions, the backward SDE has a unique square integrable solution [38, Theorem 4.3.1].

## Assumption 2.13.

(i) $\mathbb{E}\left[\Phi\left(\boldsymbol{X}^{t, \boldsymbol{\omega}}\right)^{2}\right]<\infty$, for all $\boldsymbol{\omega} \in \mathcal{W}$ and all $0 \leq t \leq s \leq T$;
(ii) $f$ is uniformly Lipshitz continuous in $(y, z)$ and continuous in $t$;
(iii) $\mathbb{E}\left[\left(\int_{0}^{T}\left|f\left(r, \boldsymbol{X}^{t, \boldsymbol{\omega}}, 0,0\right)\right| \mathrm{d} r\right)^{2}\right]$ is finite.

Condition (iii) is satisfied if, for instance, there exist $G \in \mathcal{X}$ and $g \in L^{1}(\mathbb{T})$ such that $f(r, \boldsymbol{\omega}, 0,0) \leq$ $g(r) \sqrt{\|G(\boldsymbol{\omega})\|_{\mathbb{T}}}$. We now show that

$$
U(t, \boldsymbol{\omega}):=\boldsymbol{Y}_{t}^{t, \boldsymbol{\omega}}
$$

satisfies a semi-linear path-dependent PDE.
Proposition 2.14. Let Assumptions 2.1, 2.6 and 2.13 hold. Moreover, assume the SVE (2.6) has a unique strong solution and $U \in C_{+, \alpha}^{0,2}$, with $\alpha>\frac{1}{2}-H$ and $H \in\left(0, \frac{1}{2}\right)$. Then the time derivative of $U$ exists and $U$ is a classical solution to the path-dependent PDE

$$
\begin{equation*}
\partial_{t} u(t, \boldsymbol{\omega})+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\sigma^{t, \boldsymbol{\omega}}, \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), b^{t, \boldsymbol{\omega}}\right\rangle+f\left(t, \boldsymbol{\omega}, u(t, \boldsymbol{\omega}),\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \sigma^{t, \boldsymbol{\omega}}\right\rangle\right)=0 \tag{2.8}
\end{equation*}
$$

with boundary condition $u(T, \boldsymbol{\omega})=\Phi(\boldsymbol{\omega})$.
Proof. We follow the proof of [36. Theorem 3.2] but drop one argument (the first one in $U$ ) which makes it slightly simpler.

Step 1. For any $0 \leq t \leq r \leq s \leq T$, define

$$
\begin{equation*}
\widetilde{\boldsymbol{X}}_{r, s}^{t, \boldsymbol{\omega}}:=\boldsymbol{x}_{s}+\int_{t}^{r} b\left(s, r^{\prime}, \boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r^{\prime}+\int_{t}^{r} \sigma\left(s, r^{\prime}, \boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}}\right) \mathrm{d} \boldsymbol{W}_{r^{\prime}} \tag{2.9}
\end{equation*}
$$

Hence, by strong uniqueness of (2.6), we have

$$
\boldsymbol{X}_{s}^{t, \boldsymbol{\omega}}=\boldsymbol{X}_{s}^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}, \quad \text { where } \widehat{\boldsymbol{X}}_{r^{\prime}}^{t, \boldsymbol{\omega}, r}:=\left(\boldsymbol{X}^{t, \boldsymbol{\omega}} \otimes_{r} \widetilde{\boldsymbol{X}}_{r}^{t, \boldsymbol{\omega}}\right)_{r^{\prime}}=\boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}} \mathbb{1}_{r^{\prime}<r}+\widetilde{\boldsymbol{X}}_{r, r^{\prime}}^{t, \boldsymbol{\omega}_{r^{\prime} \geq r}}
$$

Then by uniqueness of the solution to the BSDE we see that $\boldsymbol{Y}_{r}^{t, \boldsymbol{\omega}}=U\left(r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}\right)$.
Step 2. Fix $n \in \mathbb{N}, \delta=T / n, t_{i}=i \delta$ for all $i \leq n$. For $r \in\left(t_{i}, t_{i+1}\right]$, denote

$$
\boldsymbol{Y}_{r}^{n}:=U\left(t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}\right), \quad \boldsymbol{Z}_{r}^{n}:=\left\langle\partial_{\boldsymbol{\omega}} U\left(t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}\right), \sigma^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}\right\rangle
$$

Thanks to our assumptions we can apply the functional Itô formula to $U\left(t_{i+1}, \cdot\right)$ on $\left(t_{i}, t_{i+1}\right]$ :
$\mathrm{d} \boldsymbol{Y}_{r}^{n}=\left[\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} U\left(t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}\right),\left(\sigma^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}, \sigma^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} U\left(t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}\right), b^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}\right\rangle\right] \mathrm{d} r+\boldsymbol{Z}_{r}^{n} \mathrm{~d} \boldsymbol{W}_{r}$,
where we used the fact that in $(2.9), \boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}}=\widehat{\boldsymbol{X}}_{r^{\prime}}^{t, \boldsymbol{\omega}, r}$ for $r^{\prime} \in[t, r)$. Now define $\Delta \boldsymbol{Y}_{r}^{n}:=\boldsymbol{Y}_{r}^{n}-\boldsymbol{Y}_{r}^{t, \boldsymbol{\omega}}$ and $\Delta \boldsymbol{Z}_{r}^{n}:=\boldsymbol{Z}_{r}^{n}-\boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}$ which sastify a new BSDE

$$
\begin{equation*}
\mathrm{d} \Delta \boldsymbol{Y}_{r}^{n}=\widehat{f}\left(r, t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, \boldsymbol{Y}_{r}^{n}-\Delta \boldsymbol{Y}_{r}^{n}, \boldsymbol{Z}_{r}^{n}-\Delta \boldsymbol{Z}_{r}^{n}\right) \mathrm{d} r+\Delta \boldsymbol{Z}_{r}^{n} \mathrm{~d} \boldsymbol{W}_{r} \tag{2.11}
\end{equation*}
$$

where, for $r \in\left(t_{i}, t_{i+1}\right]$,

$$
\widehat{f}\left(r, t_{i+1}, \widehat{\boldsymbol{\omega}}, y, z\right):=\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}}^{2} U\left(t_{i+1}, \widehat{\boldsymbol{\omega}}\right),\left(\sigma^{r, \widehat{\boldsymbol{\omega}}}, \sigma^{r, \widehat{\boldsymbol{\omega}}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} U\left(t_{i+1}, \widehat{\boldsymbol{\omega}}\right), b^{r, \widehat{\boldsymbol{\omega}}}\right\rangle+f(r, \widehat{\boldsymbol{\omega}}, y, z)
$$

Since $\Delta \boldsymbol{Y}_{t_{i+1}}^{n}=0$, we use standard BSDE estimates [38. Theorem 4.2.1], the uniform Lipschitz continuity of $(y, z) \mapsto f(r, \boldsymbol{\omega}, y, z)$ and Jensen's inequality to get

$$
\begin{align*}
& \mathbb{E}\left[\sup _{t_{i}<r \leq t_{i+1}}\left|\Delta \boldsymbol{Y}_{r}^{n}\right|^{2}+\int_{t_{i}}^{t_{i+1}}\left|\Delta \boldsymbol{Z}_{r}^{n}\right|^{2} \mathrm{~d} r\right] \lesssim \mathbb{E}\left[\left(\int_{t_{i}}^{t_{i+1}} \widehat{f}\left(r, t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, \boldsymbol{Y}_{r}^{n}, \boldsymbol{Z}_{r}^{n}\right) \mathrm{d} r\right)^{2}\right] \\
& \lesssim \delta \mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left|\widehat{f}\left(r, t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, 0,0\right)\right|^{2}+\left|\boldsymbol{Y}_{r}^{n}\right|^{2}+\left|\boldsymbol{Z}_{r}^{n}\right|^{2} \mathrm{~d} r\right] . \tag{2.12}
\end{align*}
$$

The same estimates, this time applied to the BSDE (2.10), give

$$
\left.\left.\left.\begin{array}{l}
\mathbb{E}\left[\sup _{0 \leq r \leq T}\left|\boldsymbol{Y}_{r}^{n}\right|^{2}+\int_{0}^{T}\left|\boldsymbol{Z}_{r}^{n}\right|^{2} \mathrm{~d} r\right]  \tag{2.13}\\
\lesssim \mathbb{E}\left[\left|\Phi\left(\boldsymbol{X}^{s, \boldsymbol{\omega}}\right)\right|^{2}\right]+\mathbb{E}\left[\left(\int_{0}^{T} \widehat{f}\left(r, \frac{\lceil r N\rceil}{N}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, 0,0\right)-f(r, \widehat{\boldsymbol{X}}, \boldsymbol{\omega}, r\right.\right.
\end{array}, 0,0\right) \mathrm{~d} r\right)^{2}\right],
$$

which is finite by Assumption 2.13 (i) and $G$-growth of the pathwise derivatives. The same $G$-growth of the pathwise derivatives yields

$$
\begin{equation*}
\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left|\widehat{f}\left(r, t_{i+1}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, 0,0\right)\right|^{2} \mathrm{~d} r\right]<\infty \tag{2.14}
\end{equation*}
$$

We combine the previous estimates in (2.12), in 2.13 and in 2.14 to conclude this step:

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_{i}}^{t_{i+1}}\left|\boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}-\left\langle\partial_{\boldsymbol{\omega}} U\left(r, \widehat{\boldsymbol{X}}^{r, \boldsymbol{\omega}, r}\right), \sigma^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}\right\rangle\right|^{2} \mathrm{~d} r\right]=\mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_{i}}^{t_{i+1}}\left|\Delta \boldsymbol{Z}_{r}^{n}\right|^{2} \mathrm{~d} r\right] \\
& \leq \delta \mathbb{E}\left[\int_{0}^{T}\left|\widehat{f}\left(r, \frac{\lceil r N\rceil}{N}, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}, 0,0\right)\right|^{2}+\left|\boldsymbol{Y}_{r}^{n}\right|^{2}+\left|\boldsymbol{Z}_{r}^{n}\right|^{2} \mathrm{~d} r\right]
\end{aligned}
$$

which goes to zero as $\delta \rightarrow 0$, and therefore implies $\boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}=\left\langle\partial_{\boldsymbol{\omega}} U\left(r, \widehat{\boldsymbol{X}}^{r, \boldsymbol{\omega}, r}\right), \sigma^{r, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}}\right\rangle$.
Step 3. In particular, setting $r=t$ in (2.11) we have $\widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, t}=\boldsymbol{\omega}$ and

$$
\mathrm{d} \Delta Y_{t}^{n}=\widehat{f}\left(t, t_{i+1}, \boldsymbol{\omega}, \boldsymbol{Y}_{s}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{t}^{t, \boldsymbol{\omega}}\right) \mathrm{d} t+\Delta \boldsymbol{Z}_{t}^{n} \mathrm{~d} \boldsymbol{W}_{t}
$$

Let $t=t_{i}$, hence $t_{i+1}=t+\delta$, and recall $\Delta \boldsymbol{Y}_{t_{i+1}}^{n}=0$, therefore we have

$$
\begin{equation*}
U(t+\delta, \boldsymbol{\omega})-U(t, \boldsymbol{\omega})=-\left(\Delta \boldsymbol{Y}_{t_{i+1}}^{n}-\Delta \boldsymbol{Y}_{t_{i}}^{n}\right)=-\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}} \widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{s, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}\right) \mathrm{d} v\right]+R(\delta) \tag{2.15}
\end{equation*}
$$

where we took expectations because the left-hand-side is deterministic and where

$$
R(\delta):=\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left\{\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{s, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}\right)-\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{s, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}+\Delta \boldsymbol{Z}_{v}^{n}\right)\right\}\right.
$$

$$
\left.+\left\{\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{s, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{t, x}\right)-\widehat{f}\left(v, t_{i+1}, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{t, x}\right)\right\} \mathrm{d} v\right]
$$

We first apply Jensen's inequality and $(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)$, then Lipschitz continuity in $z$ and estimate $\sqrt{2.12)}$ for the first difference, and the regularity of $U$ and $f$ for the second (Definition 2.8(iii) as $U \in C_{+, \alpha}^{(,, 2}(\Lambda)$ and Assumption 2.13 ):

$$
\begin{aligned}
|R(\delta)|^{2} \leq & 2 \delta \mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left\{\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}\right)-\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}+\Delta \boldsymbol{Z}_{v}^{n}\right)\right\}^{2}\right. \\
& \left.+\left\{\widehat{f}\left(v, v, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}+\Delta \boldsymbol{Z}_{v}^{n}\right)-\widehat{f}\left(v, t_{i+1}, \boldsymbol{\omega}, \boldsymbol{Y}_{v}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{v}^{n}+\Delta \boldsymbol{Z}_{v}^{n}\right)\right\}^{2} \mathrm{~d} v\right] \\
= & o\left(\delta^{2}\right)
\end{aligned}
$$

Hence, dividing 2.15 by $\delta$ and taking $\delta$ to zero, the time derivative exists and is equal to

$$
-\partial_{t} U(t, \boldsymbol{\omega})=\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}}^{2} U(t, \boldsymbol{\omega}),\left(\sigma^{t, \boldsymbol{\omega}}, \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} U(t, \boldsymbol{\omega}), b^{t, \boldsymbol{\omega}}\right\rangle+f\left(t, U(t, \boldsymbol{\omega}),\left\langle\partial_{\boldsymbol{\omega}} U(t, \boldsymbol{\omega}), \sigma^{t, \boldsymbol{\omega}}\right\rangle\right)
$$

Finally, $U(T, x)=\boldsymbol{Y}_{T}^{T, x}=\Phi\left(\boldsymbol{X}^{T, \boldsymbol{\omega}}\right)=\Phi(\boldsymbol{\omega})$, and thus $U$ is a classical solution to the PPDE.
Proposition 2.15. Under the same assumptions as in Proposition 2.14, the PPDE (2.8) with its boundary condition has a unique $C_{+, \alpha}^{1,2}$ solution.

Proof. This is inspired from the proof of [29. Theorem 4.1]. In the given set of assumptions, Proposition 2.14 guarantees the existence of $u \in C_{+, \alpha}^{1,2}$, a solution to the PPDE. Then we can apply Itô's formula $\sqrt{2.5}$ to $u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)$ and, after cancelling the terms from the PPDE, we obtain

$$
\mathrm{d} u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)=-f\left(t, u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right),\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{\left.t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right\rangle}\right\rangle \mathrm{d} t+\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle \mathrm{d} \boldsymbol{W}_{t} .\right.
$$

Combined with the boundary condition and the fact that the BSDE in (2.7) has a unique solution by Assumption 2.13, this implies

$$
\left(\boldsymbol{Y}_{s}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{s}^{t, \boldsymbol{\omega}}\right)=\left(u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right),\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{\left.t, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right\rangle}\right\rangle\right)
$$

is the unique solution to the $\operatorname{BSDE}(2.7)$. In particular, $\boldsymbol{Y}_{t}^{t, \boldsymbol{\omega}}=u(t, \boldsymbol{\omega})$, concluding the proof.
Remark 2.16. In financial applications, we are interested in evaluating conditional expectations of the type $\mathbb{E}\left[\Phi(S) \mid \mathcal{F}_{t}\right]$ where $S$ is the asset price, expressed with respect to the variance. Note that in the general Volterra case (where the coefficients have feedback, or equivalently where $V$ is implicit), it is not trivial to express it as a function of $\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right)$. However, we observe that, for $0 \leq t \leq s \leq T$,

$$
\begin{align*}
\boldsymbol{X}_{s} & =\boldsymbol{X}_{0}+\int_{0}^{t}\left[b\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\sigma\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r}\right]+\int_{t}^{s}\left[b\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\sigma\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r}\right] \\
& =\boldsymbol{\Theta}_{s}^{t}+\int_{t}^{s}\left[b\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\sigma\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r}\right] . \tag{2.16}
\end{align*}
$$

Hence, by the strong uniqueness of the solutions to the SVEs (2.1) and (2.6), $\boldsymbol{X}=\boldsymbol{X}^{t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}}$ almost surely on $\mathbb{T}$. Therefore,

$$
\mathbb{E}\left[\Phi(\boldsymbol{X}) \mid \mathcal{F}_{t}\right]=\mathbb{E}\left[\Phi\left(\boldsymbol{X}^{t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}}\right) \mid \mathcal{F}_{t}\right]=\boldsymbol{Y}_{t}^{t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}}=U\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right)
$$

In particular, if the test function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ only depends on the terminal time $\boldsymbol{X}_{T}=\boldsymbol{X}^{t, \boldsymbol{\Theta}^{t}}$ then $\mathbb{E}\left[\phi\left(\boldsymbol{X}_{T}\right) \mid \mathcal{F}_{t}\right]=U\left(t, \boldsymbol{\Theta}^{t}\right)$.

Remark 2.17. The results of this section do not go through without the assumption of strong uniqueness of the solution to the SVE. This acts as a further impetus for resolving this open problem in the rough Heston model.

Remark 2.18. The functional Itô calculus introduced by Dupire 13 aimed at pricing path-dependent options on Markovian underlyings. In the present framework, the process $\boldsymbol{X}$ in (2.16) is Markovian if the coefficients are independent of $s$, and hence $\boldsymbol{\Theta}_{s}^{t}=\boldsymbol{X}_{t}$ for all $s \geq t$. The concatenated path $\boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}$ is thus frozen after $t$. Instead, Dupire stopped the path after $t$ and considered paths of different lengths which induced a different time derivative. In our setting with a Markovian process, the spatial derivative arises from perturbing a constant path in a constant direction, which boils down to perturbing it only at time $t$, as Dupire does.
2.4. Application to rough volatility. We consider the following rough volatility model:

$$
\begin{equation*}
X_{t}:=x_{0}+\int_{0}^{t} \psi\left(r, V_{r}\right) \mathrm{d} B_{r}+\zeta \int_{0}^{t} \psi\left(r, V_{r}\right)^{2} \mathrm{~d} r, \quad V_{t}:=\int_{0}^{t} K(t, r) \mathrm{d} W_{r} \tag{2.17}
\end{equation*}
$$

with $x_{0}, \zeta \in \mathbb{R}, B=\bar{\rho} \bar{W}+\rho W, W$ and $\bar{W}$ independent Brownian motions, $\rho \in[-1,1]$ and $\bar{\rho}:=\sqrt{1-\rho^{2}}$. The kernel $K: \mathbb{T}^{2} \rightarrow \mathbb{R}$ can be singular, $V$ is a Gaussian (Volterra) process and thus has finite exponential moments. We are chiefly interested in the Riemann-Liouville case $K(t, s)=(t-s)^{H-\frac{1}{2}}$. The two-dimensional $\boldsymbol{X}=(X, V)^{\top}$ is a particular case of SVE (2.1) with $d=m=2$ and

$$
b\left(t, r, x_{1}, x_{2}\right)=\left[\begin{array}{c}
\zeta \psi\left(r, x_{2}\right)^{2} \\
0
\end{array}\right], \quad \sigma\left(t, r, x_{1}, x_{2}\right)=\left[\begin{array}{cc}
\bar{\rho} \psi\left(r, x_{2}\right) & \rho \psi\left(r, x_{2}\right) \\
0 & K(t, r)
\end{array}\right]
$$

In financial models, $X$ is the $\log$ of an exponential martingale and thus $\zeta=-\frac{1}{2}$ while, for simplicity, previous papers studying weak error rates had set $\zeta=0$. In this section we check all the conditions of Proposition 2.14 and derive the pricing PPDE for the rough volatility model 2.17). In particular, this gives a rigorous justification to [35, Remark 5.2].

Denote with $\left\{\mathcal{F}_{t}\right\}_{t \in \mathbb{T}}$ the filtration generated by $(W, \bar{W})$. We note that in this case

$$
\Theta_{s}^{t}=\int_{0}^{t} K(s, r) \mathrm{d} W_{r}=\mathbb{E}\left[V_{s} \mid \mathcal{F}_{t}\right], \quad s \geq t
$$

As a special case of (2.2), this process arises from the decomposition of $V$ as, for all $0 \leq t \leq s \leq T$,

$$
V_{s}=\int_{0}^{s} K(s, r) \mathrm{d} W_{r}=\int_{0}^{t} K(s, r) \mathrm{d} W_{r}+\int_{t}^{s} K(s, r) \mathrm{d} W_{r}=: \Theta_{s}^{t}+I_{s}^{t}
$$

This decomposition is orthogonal in the sense that $I_{s}^{t}$ is independent from $\mathcal{F}_{t}$ whereas $\Theta_{s}^{t}=\mathbb{E}\left[V_{s} \mid \mathcal{F}_{t}\right]$. In particular, $\Theta_{t}^{t}=V_{t}$ and for $s \in \mathbb{T}$ fixed, $\Theta_{s}^{*}=\left(\Theta_{s}^{t}\right)_{t \in[0, s]}$ is a martingale.

Because $X$ is a semimartingale, pathwise derivatives for this component boil down to standard derivatives. Therefore the space of interest in this setting is

$$
\bar{\Lambda}=\left\{(t, x, \omega)=(t,(x, \omega)) \in \mathbb{T} \times \mathbb{R} \times \overline{\mathcal{W}}:\left.\omega\right|_{[t, T]} \in \mathcal{W}_{t}\right\}
$$

where $\mathcal{W}, \mathcal{W}_{t}, \overline{\mathcal{W}}$ are defined as in Section 2.2 , with $d=1$. We showed in the introduction, see Equation (1.3), and in a more general context in Remark 2.16, that for a function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ and $(t, x, \omega) \in \bar{\Lambda}$, conditional expectations can be expressed with a function $u: \bar{\Lambda} \rightarrow \mathbb{R}$ as $u(t, x, \omega):=$ $\mathbb{E}\left[\phi\left(X_{T}\right) \mid X_{t}=x, \Theta^{t}=\omega\right]$, which entails

$$
u\left(t, X_{t}, \Theta_{[t, T]}^{t}\right)=\mathbb{E}\left[\phi\left(X_{T}\right) \mid \mathcal{F}_{t}\right]
$$

We focus on this process for the rest of this section. Derivatives with respect to the first two components are understood as standard derivatives while the third one is the Fréchet derivative defined in (2.3).

## Assumption 2.19.

(i) The payoff function $\phi$ belongs to $C^{3}(\mathbb{R})$ and there exists $\kappa_{\phi}>0$ such that $g(x) \lesssim 1+|x|^{\kappa_{\phi}}$ for $g \in\left\{\phi, \phi^{\prime}, \phi^{\prime \prime}\right\}$ and all $x \in \mathbb{R}$;
(ii) The volatility function $\psi$ belongs to $C^{0,3}(\mathbb{T} \times \mathbb{R}$ ) (with first and second derivative in space denoted by $\psi^{\prime}$ and $\psi^{\prime \prime}$, respectively) and there exists $\kappa_{\psi}>0$ such that $g(t, x) \lesssim 1+\mathrm{e}^{\kappa_{\psi}(x+t)}$ for $g \in\left\{\psi, \psi^{\prime}, \psi^{\prime \prime}\right\}$ and all $(t, x) \in \mathbb{T} \times \mathbb{R}$.

Remark 2.20. We are limited to polynomial growth for $\phi$ because it is an open problem whether $\mathbb{E}\left[\mathrm{e}^{p X_{T}}\right]<\infty$ for most values of $(p, \zeta) \in \mathbb{R}^{2}$.

The reason for this assumption is the following lemma, whose proof can be found in Section 4.1.
Lemma 2.21. Let Assumption 2.19(ii) hold. For all $(t, x, \omega) \in \bar{\Lambda}$ and $s \in[t, T]$, we define

$$
\begin{equation*}
V_{s}^{t, \omega}:=\omega_{s}+I_{s}^{t}, \quad \text { and } \quad X_{T}^{t, x, \omega}:=x+\int_{t}^{T} \psi\left(s, V_{s}^{t, \omega}\right) \mathrm{d} B_{s}+\zeta \int_{t}^{T} \psi\left(s, V_{s}^{t, \omega}\right)^{2} \mathrm{~d} s \tag{2.18}
\end{equation*}
$$

Then, for all $p \geq 1$,

$$
\begin{equation*}
\mathbb{E}\left[\mathrm{e}^{p\|V\|_{\mathbb{T}}}\right]<\infty, \quad \sup _{s \in[t, T]} \mathbb{E}\left[\mathrm{e}^{p V_{s}^{t, \omega}}\right] \lesssim \mathrm{e}^{p \omega_{t}}, \quad \text { and } \quad \mathbb{E}\left[\left\|X^{t, x, \omega}\right\|_{\mathbb{T}}^{p}\right] \lesssim|x|^{p}+\mathrm{e}^{2 \kappa_{\psi} p\|\omega\|_{\mathbb{T}}} \tag{2.19}
\end{equation*}
$$

Therefore, any function $G: \mathbb{R}^{2} \rightarrow \mathbb{R}$ with polynomial (resp. exponential) growth in the first (resp. second) component, i.e. $G(x, v) \lesssim 1+|x|^{\ell}+\mathrm{e}^{\ell v}$ for some $\ell>0$, belongs to the set $\mathcal{X}$ corresponding to the equations in (2.17), i.e.

$$
\begin{equation*}
\mathbb{E}\left[\|G(X, V)\|_{\mathbb{T}}\right]<\infty \tag{2.20}
\end{equation*}
$$

This lemma characterises functions belonging to $\mathcal{X}$ and uses them to prove growth estimates that lead to growth estimates of the pathwise derivatives. In terms of notations, we write $\mathbb{E}_{t, x, \omega}$ for the expectation conditioned on $X_{t}=x, \Theta^{t}=\omega$. In other words, it corresponds to the expectation with $X^{t, x, \omega}$ and $V^{t, \omega}$, as in the previous lemma. The following proposition, proved in Section 4.2, gives an explicit representation of the derivatives of $u$.

Proposition 2.22. Let Assumption 2.19 hold. The function $(x, \omega) \mapsto u(t, x, \omega)$ belongs to $C_{+}^{2,2}$ and, for all $(t, x, \omega) \in \bar{\Lambda}, \eta^{(1)}, \eta^{(2)} \in \mathcal{W}_{t}$, we have

$$
\begin{aligned}
& \partial_{x} u(t, x, \omega)= \mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right)\right], \quad \partial_{x x} u(t, x, \omega)=\mathbb{E}_{t, x, \omega}\left[\phi^{\prime \prime}\left(X_{T}\right)\right] \\
&\left\langle\partial_{\omega} u(t, x, \omega), \eta\right\rangle=\mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right)\left\{\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) \eta_{s} \mathrm{~d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime}\left(s, V_{s}\right) \eta_{s} \mathrm{~d} s\right\}\right] \\
&\left\langle\partial_{\omega}\left(\partial_{x} u\right)(t, x, \omega), \eta\right\rangle=\mathbb{E}_{t, x, \omega}\left[\phi^{\prime \prime}\left(X_{T}\right)\left\{\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) \eta_{s} \mathrm{~d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime}\left(s, V_{s}\right) \eta_{s} \mathrm{~d} s\right\}\right] \\
&\left\langle\partial_{\omega \omega} u(t, x, \omega),\left(\eta^{(1)}, \eta^{(2)}\right)\right\rangle=\mathbb{E}_{t, x, \omega}\left[\phi^{\prime \prime}\left(X_{T}\right)\left\{\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime}\left(s, V_{s}\right) \eta_{s}^{(1)} \mathrm{d} s\right\}\right. \\
& \cdot\left\{\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime}\left(s, V_{s}\right) \eta_{s}^{(2)} \mathrm{d} s\right\} \\
&\left.+\phi^{\prime}\left(X_{T}\right) \int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime \prime}\left(s, V_{s}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s\right]
\end{aligned}
$$

The passage to singular kernels/directions requires to introduce some Malliavin calculus; here is a short summary of what we need in this paper. Adopting notations and definitions from [27] Section 1.2], we denote by $\mathbf{D}$ the Malliavin derivative operator with respect to $\boldsymbol{W}$ and by $\mathbb{D}^{1,2}$ its domain of application in $L^{2}(\Omega)$. For $F \in \mathbb{D}^{1,2}, \mathbf{D} F=\left(\mathrm{D}^{W} F, \mathrm{D}^{\bar{W}} F\right)^{\top}=:(\mathrm{D} F, \overline{\mathrm{D}} F)^{\top}$. The Malliavin integration by parts formula plays a crucial role

$$
\mathbb{E}\left[F \int_{0}^{T}\left\langle h_{t}, \mathrm{~d} \boldsymbol{W}_{t}\right\rangle\right]=\mathbb{E}\left[\int_{0}^{T}\left\langle\mathbf{D}_{t} F, h_{t}\right\rangle \mathrm{d} t\right]
$$

for all $F \in \mathbb{D}^{1,2}$ and $h \in L^{2}(\Omega \times \mathbb{T})$, and with $\langle\cdot, \cdot\rangle$ the inner product in $\mathbb{R}^{2}$ (not to be confused with the pathwise derivative). In particular, we use extensively the following application with $h$ a real-valued
process and $\boldsymbol{\rho}=(\rho, \bar{\rho})$

$$
\begin{equation*}
\mathbb{E}\left[F \int_{t}^{T} h_{s} \mathrm{~d} B_{s} \mid \mathcal{F}_{t}\right]=\mathbb{E}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} F, \boldsymbol{\rho}\right\rangle h_{s} \mathrm{~d} s \mid \mathcal{F}_{t}\right] . \tag{2.21}
\end{equation*}
$$

For all $n \geq 1$, the domain of application of the iterated derivatives $\mathbf{D}^{n}$ is called $\mathbb{D}^{n, 2}$ and $\mathbb{D}^{\infty, 2}:=$ $\bigcap_{n \in \mathbb{N}} \mathbb{D}^{n, 2}$ represents infinitely Malliavin differentiable random variables. For $F \in \mathbb{D}^{n, 2}$, we note that $\mathbf{D}^{n} F$ belongs to $L^{2}\left(\Omega ; L^{2}\left(\mathbb{T}^{n}\right)\right)$ or $L^{2}\left(\Omega \times \mathbb{T}^{n}\right)$ and can be seen as a stochastic process $\left\{\mathbf{D}_{\boldsymbol{s}_{n}}^{n} F ; \boldsymbol{s}_{n} \in\right.$ $\left.\mathbb{T}^{n}\right\}$. We can now state the pathwise derivative with respect to the singular kernel, whose proof is postponed to Section 4.3 .

Proposition 2.23. For all $t \in \mathbb{T}$, the conclusions of Proposition 2.22 still hold when the directions $\eta, \eta^{(1)}, \eta^{(2)} \in \mathcal{W}_{t}$ are replaced by the singular kernel $K^{t}:=K(\cdot, t)$, with the modification

$$
\begin{align*}
\left\langle\partial_{\omega}^{2} u(t, x, \omega),\left(K^{t}, K^{t}\right)\right\rangle= & \mathbb{E}_{t, x, \omega}\left[\phi^{\prime \prime}\left(X_{T}\right)\left\{\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) K(s, t) \mathrm{d} B_{s}+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime}\left(s, V_{s}\right) K(s, t) \mathrm{d} s\right\}^{2}\right]  \tag{2.22}\\
& +\mathbb{E}_{t, x, \omega}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}\right) K(s, t)^{2} \mathrm{~d} s+\zeta \int_{t}^{T}\left(\psi^{2}\right)^{\prime \prime}\left(s, V_{s}\right) K(s, t)^{2} \mathrm{~d} s\right]
\end{align*}
$$

The following proposition verifies that the derivatives satisfy the proper regularity conditions of Definition 2.8 and is proved in Section 4.4 .

Proposition 2.24. Let Assumption 2.19 hold. For all $t \in \mathbb{T}$, the function $(x, \omega) \mapsto u(t, x, \omega)$ belongs to $C_{+, \alpha}^{2,2}$ with $\alpha=1 / 2$.

As a consequence of Propositions 2.14 and 2.15, this yields the well-posedness of the PPDE, the main result of this section.
Theorem 2.25. Let Assumption 2.19 hold, then $u \in C_{+, \alpha}^{1,2,2}$, with $\alpha=1 / 2$, and is the unique classical solution of the PPDE

$$
\begin{equation*}
\partial_{t} u+\zeta \psi_{t}\left(\omega_{t}\right)^{2} \partial_{x} u+\frac{1}{2} \psi_{t}\left(\omega_{t}\right)^{2} \partial_{x x}^{2} u+\frac{1}{2}\left\langle\partial_{\omega \omega}^{2} u,\left(K^{t}, K^{t}\right)\right\rangle+\rho \psi_{t}\left(\omega_{t}\right)\left\langle\partial_{\omega}\left(\partial_{x} u\right), K^{t}\right\rangle=0 \tag{2.23}
\end{equation*}
$$

for all $(t, x, \omega) \in \bar{\Lambda}$ and with boundary condition $u(T, x, \omega)=\phi(x)$.
Proof of the Theorem. Let us start by checking that all the necessary assumptions are satisfied, namely 2.1 and 2.6 , the existence and uniqueness of strong solutions for our SDEs and 2.13. The two-dimensional SVE (2.17) is explicit and Lemma 2.21 provides the moment bounds hence Assumption 2.1 holds. Regarding the coefficients, $|\psi(y)| \leq G(y),\left|\psi^{\prime}(y)\right| \leq G(y),\left|\psi^{2}(y)\right| \leq G^{2}(y)$ and $\left|\left(\psi^{2}\right)^{\prime}(y)\right|=\left|2 \psi(y) \psi^{\prime}(y)\right| \leq$ $2 G^{2}(y)$ for some $G$ (and consequently $G^{2}$ ) in $\mathcal{X}$, from Assumption 2.19(ii) and Lemma 2.21, while $K(t, s)=$ $(t-s)^{H-\frac{1}{2}}$ and $\partial_{t} K(t, s)=\left(H-\frac{1}{2}\right)(t-s)^{H-\frac{3}{2}}$, thus Assumption 2.6 is satisfied.

Moreover, Assumption 2.13(i) is also verified because of Lemma 2.21 and Assumption 2.19, whereas Assumption 2.13(ii) and (iii) are trivially satisfied as $f \equiv 0$.

The last ingredient is Proposition 2.24 , which allows us to apply Propositions 2.14 and 2.15 . These propositions state that $u(t, \boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is seen as a two-dimensional path $\boldsymbol{\omega}=\left(\omega^{1}, \omega^{2}\right)$, is the unique solution to the PPDE (2.8)

$$
\partial_{t} u+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u,\left(\sigma^{t, \boldsymbol{\omega}}, \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} u, b^{t, \boldsymbol{\omega}}\right\rangle=0
$$

with terminal condition $u(T, \boldsymbol{\omega})=\phi\left(\omega_{T}^{1}\right)$. The dependence on $\omega^{1}$ at time $t$ is only through the point $\omega_{t}^{1}$, hence in the setting of the two-dimensional SVE (2.17), the derivatives correspond to (Remark 2.12)

$$
\begin{aligned}
\left\langle\partial_{\boldsymbol{\omega}} u, b^{t, \boldsymbol{\omega}}\right\rangle & =\left\langle\partial_{\omega^{1}} u, \zeta \psi_{t}\left(\omega_{t}^{2}\right)^{2}\right\rangle=\zeta \psi_{t}\left(\omega_{t}^{2}\right)^{2} \partial_{x} u, \\
\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u,\left(\sigma^{t, \boldsymbol{\omega}}, \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle & =\left\langle\partial_{\omega^{1} \omega^{1}} u,\left(\psi_{t}\left(\omega_{t}^{2}\right), \psi_{t}\left(\omega_{t}^{2}\right)\right)\right\rangle+2\left\langle\partial_{\omega^{1} \omega^{2}} u,\left(\rho \psi_{t}\left(\omega_{t}^{2}\right), K^{t}\right)\right\rangle+\left\langle\partial_{\omega^{2} \omega^{2}} u,\left(K^{t}, K^{t}\right)\right\rangle
\end{aligned}
$$

$$
=\psi_{t}\left(\omega_{t}^{2}\right)^{2} \partial_{x x} u+2 \rho \psi_{t}\left(\omega_{t}^{2}\right)\left\langle\partial_{\omega^{2}}\left(\partial_{x} u\right), K^{t}\right\rangle+\left\langle\partial_{\omega^{2} \omega^{2}} u,\left(K^{t}, K^{t}\right)\right\rangle
$$

This boils down to (2.23).
In passing, we showed that one can apply the functional Itô formula (2.5) to $u_{t}=u\left(t, X_{t}, \Theta^{t}\right)$, namely

$$
\begin{aligned}
\mathrm{d} u_{t}= & \left(\partial_{t} u_{t}+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u_{t},\left(\sigma^{t, \boldsymbol{\omega}}, \sigma^{t, \boldsymbol{\omega}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} u_{t}, b^{t, \boldsymbol{\omega}}\right\rangle\right) \mathrm{d} t+\left\langle\partial_{\boldsymbol{\omega}} u_{t}, \sigma^{t, \boldsymbol{\omega}}\right\rangle \mathrm{d} \boldsymbol{W}_{t} \\
= & \left(\partial_{t} u_{t}+\zeta \psi_{t}\left(V_{t}\right)^{2} \partial_{x} u_{t}+\frac{1}{2} \psi_{t}\left(V_{t}\right)^{2} \partial_{x x} u_{t}+\rho \psi_{t}\left(V_{t}\right)\left\langle\partial_{\omega^{2}}\left(\partial_{x} u_{t}\right), K^{t}\right\rangle+\frac{1}{2}\left\langle\partial_{\omega^{2} \omega^{2}} u_{t},\left(K^{t}, K^{t}\right)\right\rangle\right) \mathrm{d} t \\
& +\psi\left(V_{t}\right) \partial_{x} u_{t} \mathrm{~d} B_{t}+\left\langle\partial_{\omega^{2}} u_{t}, K^{t}\right\rangle \mathrm{d} W_{t} .
\end{aligned}
$$

## 3. Weak rates of convergence

3.1. The main result. Let $N \in \mathbb{N}$, set $\Delta_{t}:=\frac{T}{N}$. and $t_{i}:=i \Delta_{t}$ for $i=0, \ldots, N$. For conciseness we consider the model 2.17 with $\zeta=0$ and $\psi(s, v)=\psi(v)$. The latter is only cumbersome but should not alter the main results while the former generates additional terms which require a more involved analysis. In the literature, the drift has always been considered uninfluential on the rate. As a matter of fact, the rate obtained in Case 1 of our main theorem does not hold if a drift is present $(\zeta \neq 0)$. We perform an Euler discretisation of $X$ that we name $\bar{X}$ :

$$
\begin{aligned}
& \bar{X}_{t_{0}}=x_{0} \\
& \bar{X}_{t_{i+1}}=\bar{X}_{t_{i}}+\psi\left(V_{t_{i}}\right)\left(B_{t_{i+1}}-B_{t_{i}}\right)
\end{aligned}
$$

for $i=0, \ldots, N-1$, where $V$ is simulated exactly (e.g. by Cholesky decomposition). Then, we extend to the whole time interval the process $\bar{X}$ by interpolation

$$
\begin{equation*}
\bar{X}_{t}=x_{0}+\int_{0}^{t} \psi\left(V_{\kappa_{s}}\right) \mathrm{d} B_{s} \tag{3.1}
\end{equation*}
$$

with $\kappa_{s}=t_{i}$ for $s \in\left[t_{i}, t_{i+1}\right)$. Our goal is to estimate the difference between the price and and its approximation with respect to the nmber of grid points $N$ :

$$
\mathcal{E}^{N}:=\mathbb{E}\left[\phi\left(X_{T}\right)\right]-\mathbb{E}\left[\phi\left(\bar{X}_{T}\right)\right]
$$

Our main result characterises the weak rate of convergence this numerical scheme as follows:

## Theorem 3.1.

Case 1. If $\phi$ is quadratic, $\psi$ has exponential growth and there exist $\gamma, \overline{\mathbf{C}}>0$ independent of $t, t_{i}$ such that

$$
\begin{equation*}
\left|\mathbb{E}\left[g\left(V_{t_{i}}\right)-g\left(V_{t}\right)\right]\right| \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right), \quad \text { for } g \in\left\{\psi, \psi^{2}\right\}, t \in\left[t_{i}, t_{i+1}\right) \tag{3.2}
\end{equation*}
$$

then $\mathcal{E}^{N}=\mathcal{O}\left(N^{-1}\right)$.
Case 2. If $\phi, \psi \in C^{\infty}, \phi$ and its derivatives have polynomial growth, while $\psi$ and its derivatives have exponential growth, and there exist $\gamma, \overline{\mathbf{C}}>0$ independent of $t, t_{i}$ such that for all $g \in$ $\left\{\psi^{(n)}, \psi^{2(n)}\right\}_{n \geq 0}, t \in\left[t_{i}, t_{i+1}\right)$, either condition holds

$$
\begin{array}{ll}
\text { (i) }\left|\mathbb{E}\left[g\left(V_{t_{i}}\right)-g\left(V_{t}\right)\right]\right| & \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \\
\text { (ii) }\left|\mathbb{E}\left[g\left(V_{t_{i}}\right)-g\left(V_{t}\right)\right]\right| & \leq \overline{\mathbf{C}}\left(t-t_{i}\right)^{H+\frac{1}{2}} \tag{3.3}
\end{array}
$$

then $\mathcal{E}^{N}=\mathcal{O}\left(N^{-H-\frac{1}{2}}\right)$.
Remark 3.2. The growth assumptions ensure that $\phi^{(n)}\left(X_{T}\right), \psi^{(n)}\left(V_{t}\right), \psi^{2(n)}\left(V_{t}\right)$ are bounded in $L^{p}$ for all $p \geq 1$ by Lemma 2.21 .

Example 3.3. Both conditions (3.2) and (3.3) are satisfied if $\psi$ is polynomial or exponential. Indeed, since $V_{t}$ is centered Gaussian with variance $\frac{1}{2 H} t^{2 H}$, then, for $p \geq 1$ even, $\mathbb{E}\left[V_{t}^{p}\right]=\frac{(p-1)!!}{(2 H)^{p / 2}} t^{H p}$ and $\mathbb{E}\left[\mathrm{e}^{\nu V_{t}}-\mathrm{e}^{\nu V_{t_{i}}}\right]=\exp \left\{\frac{\nu^{2} t^{2 H}}{4 H}\right\}-\exp \left\{\frac{\nu^{2} t_{i}^{2 H}}{4 H}\right\} \leq C\left(t^{2 H}-t_{i}^{2 H}\right)$ for some $C>0$; furthermore, if $\psi$ is polynomial or exponential, so are $\psi^{2}$ and all their derivatives.

It may seem unintuitive that, for any positive value of $\gamma$, conditions $(3.2)$ and (3.3) (i) yields a rate higher than $\gamma$. It turns out that this local error amounts to a global error of order one after integrating over $t$ and sum over $i$, as demonstrated by Lemma 3.7.

For clarity, the proof of the Theorem 3.1, and consequently of this section, is decomposed into several steps:
i) Section 3.2 gathers some preliminary results. We first break down the error $\mathcal{E}^{N}$ into a sum of convenient terms in Proposition 3.4. Then we prove a joint chaos expansion for products in Lemma 3.5.
ii) Section 3.3 focuses on Case 1, which is rather straightforward, yet enlightening and helpful to understand the underlying idea exploited in Case 2.
iii) Section 3.4 onwards focuses on Case 2:

- Each term of the sum is a difference which we carefully express via its joint chaos expansion to avoid the strong error;
- This new formula hinges on multiple Malliavin derivatives which require new notations, introduced in Section 5.1.
- The estimates required for the proof of Case 2 are gathered in Proposition 3.10. The details are more involved and hence are postponed to Sections 5 and 6 .
- The convergence of the expansion is the subject of Proposition 3.11, proved in Section 7 .
3.2. Decomposition of the error. We first express the error into a convenient sum, which is key to the main proof, in the spirit of [6].

Proposition 3.4. Let $\bar{u}_{t}:=u\left(t, \bar{X}_{t}, \Theta_{[t, T]}^{t}\right)$. Under Assumption 2.19, we can write

$$
\mathcal{E}^{N}=\sum_{i=0}^{N-1} \mathfrak{A}_{i}
$$

where

$$
\mathfrak{A}_{i}:=\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left\{\frac{1}{2}\left(\psi\left(V_{t}\right)^{2}-\psi\left(V_{t_{i}}\right)^{2}\right) \partial_{x x}^{2} \bar{u}_{t}+\rho\left(\psi\left(V_{t}\right)-\psi\left(V_{t_{i}}\right)\right)\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right\} \mathrm{d} t\right],
$$

with
$\partial_{x x}^{2} \bar{u}_{t}=\mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}\right) \mid X_{t}=\bar{X}_{t}, \Theta^{t}\right] \quad$ and $\quad\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle=\mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}\right) \int_{t}^{T} \psi^{\prime}\left(V_{s}\right) K(s, t) \mathrm{d} B_{s} \mid X_{t}=\bar{X}_{t}, \Theta^{t}\right]$.

This formulation for the derivatives is equivalent to

$$
\partial_{x x}^{2} \bar{u}_{t}=\mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \mid \mathcal{F}_{t}\right]
$$

where $X_{T}^{t, \bar{X}_{t}}:=\bar{X}_{t}+\int_{t}^{T} \psi\left(V_{s}\right) \mathrm{d} B_{s}$, and similarly for $\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle$.
Proof. Recalling that $t_{0}=0$ and $t_{N}=T$, we write $\mathcal{E}^{N}$ as a telescopic sum:

$$
\begin{aligned}
\mathbb{E}\left[\phi\left(X_{T}\right)-\phi\left(\bar{X}_{T}\right)\right] & =\mathbb{E}\left[\mathbb{E}\left[\phi\left(X_{T}\right) \mid X_{0}=\bar{X}_{0}\right]\right]-\mathbb{E}\left[\mathbb{E}\left[\phi\left(\bar{X}_{T}\right) \mid X_{T}=\bar{X}_{T}\right]\right] \\
& =\mathbb{E}\left[u\left(t_{0}, \bar{X}_{t_{0}}, \Theta_{\left[t_{0}, t_{N}\right]}^{t_{0}}\right)\right]-\mathbb{E}\left[u\left(t_{n}, \bar{X}_{t_{N}}, \Theta_{t_{N}}^{t_{N}}\right)\right] \\
& =\sum_{i=0}^{N-1}\left\{\mathbb{E}\left[u\left(t_{i}, \bar{X}_{t_{i}}, \Theta_{\left[t_{i}, t_{N}\right]}^{t_{i}}\right)\right]-\mathbb{E}\left[u\left(t_{i+1}, \bar{X}_{t_{i+1}}, \Theta_{\left[t_{i+1}, t_{N}\right]}^{t_{i+1}}\right)\right]\right\}=: \sum_{i=0}^{N-1} \mathfrak{A}_{i} .
\end{aligned}
$$

We now show that the terms $\left(\mathfrak{A}_{i}\right)_{i=0, \ldots, N-1}$ have the form given in the proposition. By virtue of the representation (3.1) and the regularity of $u$ from Proposition (2.25), we can apply the functional Itô formula 2.5 on $\left[t_{i}, t_{i+1}\right)$ as

$$
\begin{aligned}
\mathrm{d} \bar{u}_{t}= & \left(\partial_{t} \bar{u}_{t}+\psi\left(V_{t_{i}}\right)^{2} \frac{1}{2} \partial_{x x}^{2} \bar{u}_{t}+\frac{1}{2}\left\langle\partial_{\omega \omega}^{2} \bar{u}_{t},\left(K^{t}, K^{t}\right)\right\rangle+\rho \psi\left(V_{t_{i}}\right)\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right) \mathrm{d} t \\
& +\psi\left(V_{t_{i}}\right) \partial_{x} \bar{u}_{t} \mathrm{~d} B_{t}+\left\langle\partial_{\omega} \bar{u}_{t}, K^{t}\right\rangle \mathrm{d} W_{t}
\end{aligned}
$$

Combining this with the path-dependent $\operatorname{PDE}(2.23$ with $\zeta=0$, we obtain

$$
\begin{aligned}
\mathfrak{A}_{i} & =\mathbb{E}\left[u\left(t_{i}, \bar{X}_{t_{i}}, \Theta_{\left[t_{i}, T\right]}^{t_{i}}\right)-u\left(t_{i+1}, \bar{X}_{t_{i+1}}, \Theta_{\left[t_{i+1}, T\right]}^{t_{i+1}}\right)\right] \\
& =-\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left\{\partial_{t} \bar{u}_{t}+\psi\left(V_{t_{i}}\right)^{2} \frac{1}{2} \partial_{x x}^{2} \bar{u}_{t}+\frac{1}{2}\left\langle\partial_{\omega \omega}^{2} \bar{u}_{t},\left(K^{t}, K^{t}\right)\right\rangle+\rho \psi\left(V_{t_{i}}\right)\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right\} \mathrm{d} t\right] \\
& =\mathbb{E}\left[\int_{t_{i}}^{t_{i+1}}\left\{\left(\psi\left(V_{t}\right)^{2}-\psi\left(V_{t_{i}}\right)^{2}\right) \frac{1}{2} \partial_{x x}^{2} \bar{u}_{t}+\rho\left(\psi\left(V_{t}\right)-\psi\left(V_{t_{i}}\right)\right)\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right\} \mathrm{d} t\right],
\end{aligned}
$$

as claimed.
We must find a way to express the differences without invoking the strong rate of convergence. More precisely, we want to exploit our assumptions, which require to separate the function of $V$ and the derivative of $\bar{u}$ into two distinct expectations. Leveraging on the Wiener chaos expansion we obtain the following representation:

Lemma 3.5. For $g \in C^{\infty}(\mathbb{R})$ such that $g$ and its derivatives have at most exponential growth and $F \in$ $\mathbb{D}^{\infty, 2}$, we have

$$
\begin{equation*}
\mathbb{E}\left[F g\left(V_{t}\right)\right]=\mathbb{E}[F] \mathbb{E}\left[g\left(V_{t}\right)\right]+\sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0, t]^{n}} \mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} F\right] \mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} g\left(V_{t}\right)\right] \mathrm{d} \boldsymbol{s}_{n} \tag{3.5}
\end{equation*}
$$

where $\boldsymbol{s}_{n}=\left(s_{1}, \cdots, s_{n}\right)$ and $\mathrm{d} \boldsymbol{s}_{n}:=\mathrm{d} s_{1} \cdots \mathrm{~d} s_{n}$.
Recall that D is the Malliavin derivative with respect to $W$ only.
Proof. Start with the Wiener chaos expansion of $g\left(V_{t}\right)$ [27, Theorem 1.1.2]: for all $t \in \mathbb{T}, n \in \mathbb{N}$, there exist symmetric deterministic functions $f_{n}^{t} \in L^{2}\left([0, t]^{n}\right)$ such that

$$
g\left(V_{t}\right)=\mathbb{E}\left[g\left(V_{t}\right)\right]+\sum_{n=1}^{\infty} I_{n}\left(f_{n}^{t}\right)
$$

where, for any $f \in L^{2}\left([0, t]^{n}\right)$ symmetric,

$$
I_{n}(f)=\int_{[0, t]^{n}} f\left(\boldsymbol{s}_{n}\right) W\left(\mathrm{~d} s_{1}\right) \cdots W\left(\mathrm{~d} s_{n}\right)=n!\int_{0}^{t} \int_{0}^{t_{n}} \cdots \int_{0}^{t_{2}} f\left(\boldsymbol{s}_{n}\right) \mathrm{d} W_{s_{1}} \cdots \mathrm{~d} W_{s_{n}}
$$

The first integral is a multiple Wiener integral and the second is a multiple Itô integral. For $F \in \mathbb{D}^{n}$, $n$ applications of the Malliavin integration by parts yield

$$
\mathbb{E}\left[F I_{n}\left(f_{n}^{t}\right)\right]=\int_{[0, t]^{n}} \mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} F\right] f_{n}^{t}\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}
$$

Finally, the coefficients $\left(f_{n}^{t}\right)_{n \geq 1}$ can be formulated [27] Exercise 1.2.6] as

$$
f_{n}^{t}\left(\boldsymbol{s}_{n}\right)=\frac{1}{n!} \mathbb{E}\left[\mathrm{D}_{s_{n}}^{n} g\left(V_{t}\right)\right]
$$

where the Malliavin derivative is taken with respect to $W$ only.
Remark 3.6. This result naturally extends if one replaces $g\left(V_{t}\right)$ by $g \in \mathbb{D}^{\infty, 2}$. Moreover, an alternative proof consists in writing the chaos expansion of both $F$ and $G$ and noticing that, by the orthogonality of the Wiener chaos and Itô's isometry, $\mathbb{E}\left[I_{n}\left(f_{n}^{F}\right) I_{m}\left(f_{m}^{G}\right)\right]=\mathbb{1}_{n=m} \int_{[0, t]^{n}} f_{n}^{F}\left(s_{n}\right) f_{n}^{G}\left(s_{n}\right) \mathrm{d} \boldsymbol{s}_{n}$, where $f_{n}^{F}$ and $f_{n}^{G}$ are the deterministic functions appearing in the chaos expansions of $F$ and $G$ respectively.
3.3. Proof of Theorem 3.1-Case 1. In the standard approach, the local weak error rate of the integrand in $\mathfrak{A}_{i}$ is the global rate of $\mathcal{E}^{N}$. Thanks to the following observation, we can obtain rate one even when the weak local error $t^{\gamma}-t_{i}^{\gamma}$ would yield a lower rate.
Lemma 3.7. For any $\gamma>0$,

$$
\sum_{i=0}^{N-1} \int_{t_{i}}^{t_{i+1}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathrm{d} t \leq T^{1+\gamma} \Delta_{t}
$$

Case 1 of Theorem 3.1 (quadratic payoff) follows immediately. Indeed, for all $t \geq 0, g \in\left\{\psi, \psi^{2}\right\}$ and $\partial \bar{u}_{t} \in\left\{\partial_{x x} \bar{u}_{t},\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right\}$, if $\phi$ is quadratic then $\mathrm{D}_{s} \partial \bar{u}_{t}=0$ and 3.5 reads

$$
\begin{equation*}
\mathbb{E}\left[g\left(V_{t}\right) \partial \bar{u}_{t}\right]=\mathbb{E}\left[g\left(V_{t}\right)\right] \mathbb{E}\left[\partial \bar{u}_{t}\right] \tag{3.6}
\end{equation*}
$$

By assumption there exist $\gamma, \overline{\mathbf{C}}>0$ such that $\left|\mathbb{E}\left[g\left(V_{t_{i}}\right)-g\left(V_{t}\right)\right]\right| \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right)$ for $g \in\left\{\psi, \psi^{2}\right\}$. Lemma 3.7 thus applies with $\gamma=2 H$ and Case 1 follows, yielding a rate of order one.
Remark 3.8. If a drift is present $(\zeta \neq 0)$, then $(3.6)$ is not verified and the conclusion does not hold.
Proof of Lemma 3.7. The following computations are straightforward:

$$
\begin{aligned}
\sum_{i=0}^{N-1} \int_{t_{i}}^{t_{i+1}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathrm{d} t & =\sum_{i=0}^{N-1}\left\{\frac{t_{i+1}^{\gamma+1}-t_{i}^{\gamma+1}}{\gamma+1}-t_{i}^{\gamma}\left(t_{i+1}-t_{i}\right)\right\} \\
& =\sum_{i=0}^{N-1}\left\{\frac{\left((i+1) \Delta_{t}\right)^{\gamma+1}-\left(i \Delta_{t}\right)^{\gamma+1}}{\gamma+1}-\left(i \Delta_{t}\right)^{\gamma} \Delta_{t}\right\} \\
& =\Delta_{t}^{\gamma+1}\left\{\frac{1}{\gamma+1} \sum_{i=0}^{N-1}\left[(i+1)^{\gamma+1}-i^{\gamma+1}\right]-\sum_{i=0}^{N-1} i^{\gamma}\right\} \\
& \leq \Delta_{t}^{\gamma+1}\left\{\frac{N^{\gamma+1}}{\gamma+1}-\int_{0}^{N-1} x^{\gamma} \mathrm{d} x\right\}=\frac{\Delta_{t}^{\gamma+1}}{\gamma+1}\left(N^{\gamma+1}-(N-1)^{\gamma+1}\right) \\
& \leq \frac{\Delta_{t}^{\gamma+1}}{\gamma+1} \max _{x \in[N-1, N]} x^{\gamma}(\gamma+1)(N-(N-1))=\Delta_{t}^{\gamma+1} N^{\gamma}=\Delta_{t} T^{\gamma+1}
\end{aligned}
$$

where the last line follows from the mean value theorem and the monotonicity of $x \mapsto x^{\gamma}$.
3.4. Proof of Theorem 3.1-Case 2. Due to the large number of terms we need to name and estimate, we adopt the following rule: each important term is named as a capital letter from the mathfrak alphabet, starting with A " $\mathfrak{A}$ " above. That way we go further down in the alphabet when decomposing our formula and back up to $\mathfrak{A}$ when putting all the estimates together. Section 5 goes from A to H (omitting E"E" because it looks like C "C"); Section 6 from J to T (omitting O" because it looks like Q" $\mathfrak{Q}$ ").

We fix an interval $\left[t_{i}, t_{i+1}\right.$ ), so that we omit the superscript " $i$ " in the variables from now on. From Proposition 3.4, we can write

$$
\mathfrak{A}_{i}=\frac{1}{2} \int_{t_{i}}^{t_{i+1}} \underbrace{\mathbb{E}\left[\left(\psi\left(V_{t}\right)^{2}-\psi\left(V_{t_{i}}\right)^{2}\right) \partial_{x x}^{2} \bar{u}_{t}\right]}_{\mathfrak{B}_{1}(t)} \mathrm{d} t+\rho \int_{t_{i}}^{t_{i+1}} \underbrace{\mathbb{E}\left[\left(\psi\left(V_{t}\right)-\psi\left(V_{t_{i}}\right)\right)\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right]}_{\mathfrak{B}_{2}(t)} \mathrm{d} t
$$

where the derivatives are expressed in (3.4). By the chaos expansion in Lemma 3.5 .

$$
\begin{align*}
& \mathfrak{B}_{1}(t)=: \mathfrak{C}_{1}^{0}(t)+\sum_{n=1}^{\infty} \underbrace{\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \mathfrak{D}_{1}^{n 1}\left(\boldsymbol{s}_{n}\right) \mathfrak{D}_{1}^{n 2}\left(\boldsymbol{s}_{n}\right)}_{\mathfrak{C}_{1}^{n}(t)} \mathrm{d} \boldsymbol{s}_{n} \\
& \mathfrak{B}_{2}(t)=: \mathfrak{C}_{2}^{0}(t)+\sum_{n=1}^{\infty} \underbrace{\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \mathfrak{D}_{2}^{n 1}\left(\boldsymbol{s}_{n}\right) \mathfrak{D}_{2}^{n 2}\left(\boldsymbol{s}_{n}\right)}_{\mathfrak{C}_{2}^{n}(t)} \mathrm{d} \boldsymbol{s}_{n} \tag{3.7}
\end{align*}
$$

where all the terms are defined as (we drop the $\boldsymbol{s}_{n}$ notation in the $\mathfrak{D}$ terms for clarity)

$$
\begin{gathered}
\mathfrak{C}_{1}^{0}(t):=\mathbb{E}\left[\psi\left(V_{t}\right)^{2}-\psi\left(V_{t_{i}}\right)^{2}\right] \mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right)\right], \\
\mathfrak{C}_{2}^{0}(t):=\mathbb{E}\left[\psi\left(V_{t}\right)-\psi\left(V_{t_{i}}\right)\right] \mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \int_{t}^{T} \psi^{\prime}\left(V_{s}\right) K(s, t) \mathrm{d} B_{s}\right], \\
\mathfrak{D}_{1}^{n 1}:=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}}\left(\psi\left(V_{t}\right)^{2}-\psi\left(V_{t_{i}}\right)^{2}\right)\right], \quad \mathfrak{D}_{2}^{n 1}:=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}}\left(\psi\left(V_{t}\right)-\psi\left(V_{t_{i}}\right)\right)\right], \\
\mathfrak{D}_{1}^{n 2}:=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right)\right], \quad \mathfrak{D}_{2}^{n 2} \quad:=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \int_{t}^{T} \psi^{\prime}\left(V_{s}\right) K(s, t) \mathrm{d} B_{s}\right] .
\end{gathered}
$$

Remark 3.9. The terms $\mathfrak{C}_{1}^{0}(t)$ and $\mathfrak{C}_{2}^{0}(t)$ can be treated following the proof of Theorem 3.1-Case 1, and bring an error contribution of order 1.

The following proposition summarises the different bounds required for the general convergence rate of the scheme, and is analysed-and proved-more in details later on.
Proposition 3.10. The following bounds hold for any $t \in\left[t_{i}, t_{i+1}\right)$, with the constants detailed in Appendix $B$ all depend on $T, H, \phi, \psi$ :

$$
\begin{aligned}
& \left|\mathfrak{D}_{1}^{n 1}\right| \leq \overline{\mathfrak{D}_{1}^{n 1}}\left(t, t_{i}, s_{n}\right), \quad \text { [Section 5.2], } \\
& \left|\mathfrak{D}_{2}^{n 1}\right| \leq \overline{\mathfrak{D}_{2}^{n 1}}\left(t, t_{i}, s_{n}\right), \quad \text { [Section 5.2], } \\
& \left|\mathfrak{D}_{1}^{n 2}\right| \leq \overline{\mathfrak{D}_{1}^{n 2}}\left(t, t_{i}, \boldsymbol{s}_{n}\right), \quad[\text { Section 5.3], } \\
& \left|\mathfrak{D}_{2}^{n 2}\right| \leq \overline{\mathfrak{D}_{2}^{n 2}}\left(t, t_{i}, \boldsymbol{s}_{n}\right), \quad \text { [Section5.4], } \\
& \left|\mathfrak{C}_{1}^{n}(t)\right| \leq\left(t^{\gamma}-t_{i}^{\gamma}\right) \overline{\overline{\mathcal{M}_{1}^{n}}}+{ }_{1} \overline{\overline{\mathfrak{N}_{1}^{n}}} \Delta_{t}^{H_{+}}+{ }_{2} \overline{\overline{\mathfrak{N}_{1}^{n}}} \Delta_{t}^{3 H+\frac{1}{2}}, \quad \text { [Section 6.1], } \\
& \left|\mathfrak{C}_{2}^{n}(t)\right| \leq\left(t^{\gamma}-t_{i}^{\gamma}\right) \overline{\overline{\mathcal{M}_{2}^{n}}}+{ }_{1} \overline{\overline{\mathfrak{N}_{2}^{n}}} \Delta_{t}^{H_{+}}+{ }_{2} \overline{\overline{\mathfrak{N}_{2}^{n}}} \Delta_{t}^{3 H+\frac{1}{2}} \text {, [Section 6.2]. }
\end{aligned}
$$

The proofs of the $\mathfrak{D}$ estimates, presented in Section 5 consist in computing multiple Malliavin derivatives and isolating the kernels coming out from $\mathrm{D}_{s_{n}} V_{t}$ and $\mathrm{D}_{s_{n}} X_{T}^{t, \overline{X_{t}}}$. Section 6 deals with the proof of the $\mathfrak{C}$ estimates by careful integration. We prove in Section 7 that the series in (3.7) converge with the following estimate.

Proposition 3.11. For any $t \in\left[t_{i}, t_{i+1}\right)$, there exist $\overline{\mathfrak{B}}_{0}, \overline{\mathfrak{B}}_{1}, \overline{\mathfrak{B}}_{2}>0$ such that

$$
\frac{1}{2}\left|\mathfrak{B}_{1}(t)\right|+\left|\mathfrak{B}_{2}(t)\right| \leq \overline{\mathfrak{B}}_{0} \Delta_{t}^{H_{+}}+\overline{\mathfrak{B}}_{1}\left(t^{\gamma}-t_{i}^{\gamma}\right)+\overline{\mathfrak{B}}_{2} \Delta_{t}^{3 H+\frac{1}{2}}
$$

The constants $\overline{\mathfrak{B}}_{0}, \overline{\mathfrak{B}}_{1}, \overline{\mathfrak{B}}_{2}$ can be deduced easily in closed form from all the others, but we omit this straightforward detail here. This is sufficient to finish the proof of the main result, exploiting Lemma 3.7.

$$
\mathcal{E}^{N}=\sum_{i=0}^{N-1} \mathfrak{A}_{i}=\sum_{i=1}^{N-1} \int_{t_{i}}^{t_{i+1}}\left(\frac{1}{2} \mathfrak{B}_{1}(t)+\mathfrak{B}_{2}(t)\right) \mathrm{d} t \leq T\left(\overline{\mathfrak{B}}_{0} \Delta_{t}^{H_{+}}+T^{\gamma} \overline{\mathfrak{B}}_{1} \Delta_{t}+\overline{\mathfrak{B}}_{2} \Delta_{t}^{3 H+\frac{1}{2}}\right)
$$

and the order of convergence of the numerical method is thus $H+\frac{1}{2}$.

## 4. Proofs of Section 2.4

4.1. Proof of Lemma 2.21. Let $p>0$. On the one hand, $V$ is a Gaussian process hence $\mathbb{E}\left[\mathrm{e}^{\left.p\|V\|_{\mathbb{T}}\right]}\right.$ is finite [23, Lemma 6.13], which proves the first bound of (2.19). On the other hand, by Gaussian computations,

$$
\mathbb{E}\left[\mathrm{e}^{p V_{s}^{t, \omega}}\right]=\mathrm{e}^{p \omega_{t}} \mathbb{E}\left[\exp \left(p \int_{t}^{s} K(t, r) \mathrm{d} W_{r}\right)\right]=\exp \left(p \omega_{t}+\frac{p^{2}(s-t)^{2 H}}{4 H}\right)
$$

yielding the second estimate. For the last one, by BDG and Jensen's inequalities, we have

$$
\mathbb{E}\left[\sup _{s \in[t, T]}\left|X_{s}^{t, x, \omega}\right|^{p}\right] \leq 3^{p-1} \mathbb{E}\left[|x|^{p}+\sup _{s \in[t, T]}\left(\int_{t}^{s} \psi\left(r, V_{r}^{t, \omega}\right) \mathrm{d} B_{r}\right)^{p}+\zeta^{p}\left(\int_{0}^{T} \psi\left(r, V_{r}^{t, \omega}\right)^{2} \mathrm{~d} r\right)^{p}\right]
$$

$$
\begin{aligned}
& \leq 3^{p-1} \mathbb{E}\left[|x|^{p}+\mathfrak{b}_{p}\left(\int_{t}^{T} \psi\left(r, V_{r}^{t, \omega}\right)^{2} \mathrm{~d} r\right)^{p / 2}+\zeta^{p} T^{p-1} \int_{t}^{T}\left|\psi\left(r, V_{r}^{t, \omega}\right)\right|^{2 p} \mathrm{~d} r\right] \\
& \leq 3^{p-1} \mathbb{E}\left[|x|^{p}+\mathfrak{b}_{p} T^{p / 2-1} \int_{0}^{T}\left|\psi\left(r, V_{r}\right)\right|^{p} \mathrm{~d} s+\zeta^{p} T^{p-1} \int_{0}^{T}\left|\psi\left(r, V_{r}\right)\right|^{2 p} \mathrm{~d} r\right]
\end{aligned}
$$

where $\mathfrak{b}_{p}$ is the BDG constant and in the last line we have exploited that Assumption 2.19 (ii) gives

$$
\mathbb{E}\left[\int_{0}^{T}\left|\psi\left(r, V_{r}\right)\right|^{2 p} \mathrm{~d} r\right] \leq \sup _{r \in[0, T]} \mathbb{E}\left[\mathrm{e}^{2 p \kappa_{\psi}\left(r+V_{r}^{t, \omega}\right)}\right]
$$

This, combined with the second estimate, yields the claim. By assumption there exists $\ell>0$ such that $G\left(X_{t}, V_{t}\right) \lesssim 1+\left|X_{t}\right|^{\ell}+\mathrm{e}^{\ell V_{t}}$. It then follows from 2.19 that $\mathbb{E}\left[\|G(X, V)\|_{\mathbb{T}}\right]$ is finite.
4.2. Proof of Proposition 2.22. ${ }^{1}$ For all $(s,(t, x, \omega)) \in \mathbb{T} \times \bar{\Lambda}$, such that $t<s$, recall $V_{s}^{t, \omega}$ and $X_{T}^{t, x, \omega}$ defined in 2.18).

- We start by considering the derivatives in $x$ and for clarity we note $X_{T}^{x}=X_{T}^{t, x, \omega}$, with $(t, \omega)$ fixed. For any $\varepsilon>0$, notice that $X_{T}^{x+\varepsilon}-X_{T}^{x}=\varepsilon$ hence

$$
\phi\left(X_{T}^{x+\varepsilon}\right)-\phi\left(X_{T}^{x}\right)=\left(X_{T}^{x+\varepsilon}-X_{T}^{x}\right) \int_{0}^{1} \phi^{\prime}\left(\lambda X_{T}^{x+\varepsilon}+(1-\lambda) X_{T}^{x}\right) \mathrm{d} \lambda=\varepsilon \int_{0}^{1} \phi^{\prime}\left(X_{T}^{x}+\lambda \varepsilon\right) \mathrm{d} \lambda
$$

Since $\phi^{\prime} \in C^{1}$ it is also locally Lipschitz continuous which entails

$$
\begin{aligned}
\mathbb{E}_{t, \omega}\left[\frac{\phi\left(X_{T}^{x+\varepsilon}\right)-\phi\left(X_{T}^{x}\right)}{\varepsilon}-\phi^{\prime}\left(X_{T}^{x}\right)\right] & =\mathbb{E}_{t, \omega}\left[\int_{0}^{1} \phi^{\prime}\left(X_{T}^{x}+\lambda \varepsilon\right)-\phi^{\prime}\left(X_{T}^{x}\right) \mathrm{d} \lambda\right] \\
& \leq \mathbb{E}_{t, \omega}\left[\sup _{\varepsilon^{\prime} \in(0, \varepsilon)}\left|\phi^{\prime \prime}\left(X_{T}^{x+\varepsilon^{\prime}}\right)\right|\right] \int_{0}^{1} \lambda \varepsilon \mathrm{~d} \lambda,
\end{aligned}
$$

which tends to zero as $\varepsilon \rightarrow 0$ by Assumption 2.19 (i) and 2.19 , as

$$
\begin{aligned}
\mathbb{E}_{t, \omega}\left[\sup _{\varepsilon^{\prime} \in(0, \varepsilon)}\left|\phi^{\prime \prime}\left(X_{T}^{x+\varepsilon^{\prime}}\right)\right|\right] & =\mathbb{E}_{t, \omega}\left[\sup _{\varepsilon^{\prime} \in(0, \varepsilon)}\left|\phi^{\prime \prime}\left(X_{T}^{x}+\varepsilon^{\prime}\right)\right|\right] \lesssim \mathbb{E}_{t, \omega}\left[\sup _{\varepsilon^{\prime} \in(0, \varepsilon)}\left(1+\left|X_{T}^{x}+\varepsilon^{\prime}\right|^{\kappa_{\phi}}\right)\right] \\
& \lesssim 1+\sup _{\varepsilon^{\prime} \in(0, \varepsilon)}\left(\varepsilon^{\prime}\right)^{\kappa_{\phi}}+\mathbb{E}_{t, \omega}\left[\left|X_{T}^{x}\right|^{\kappa_{\phi}}\right]<\infty
\end{aligned}
$$

This proves the expression of the first derivative. The second space derivative is proved in the same fashion since $\psi^{\prime \prime}$ also belongs to $C^{1}$ by assumption.

- We turn to the pathwise derivative with respect to $\eta \in \mathcal{W}_{t}$. We fix $(t, x)$ and write (2.18) as $X_{T}^{\omega}$. For clarity, we only show the proof in the case $\zeta=0$, the general case being analogous. Recall the definition of the pathwise derivative in (2.3). Similarly as before, for any $\varepsilon>0$, we write

$$
\begin{equation*}
\phi\left(X_{T}^{\omega+\varepsilon \eta}\right)-\phi\left(X_{T}^{\omega}\right)=\Delta_{\varepsilon \eta} X_{T}^{\omega} \widetilde{\phi}(\varepsilon) \tag{4.1}
\end{equation*}
$$

where $\Delta_{\varepsilon \eta} X_{T}^{\omega}:=X_{T}^{\omega+\varepsilon \eta}-X_{T}^{\omega}$ and $\widetilde{\phi}(\varepsilon):=\int_{0}^{1} \phi^{\prime}\left(X_{T}^{\omega}+\lambda \Delta_{\varepsilon \eta} X_{T}^{\omega}\right) \mathrm{d} \lambda$. Therefore,

$$
\begin{aligned}
& \mathbb{E}_{t, x}\left[\frac{\phi\left(X_{T}^{\omega+\varepsilon \eta}\right)-\phi\left(X_{T}^{\omega}\right)}{\varepsilon}-\phi^{\prime}\left(X_{T}^{\omega}\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right] \\
& \left.=\mathbb{E}_{t, x}\left[\widetilde{\phi}(\varepsilon)\left(\frac{\Delta_{\varepsilon \eta} X_{T}^{\omega}}{\varepsilon}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right)\right]+\mathbb{E}_{t, x}\left[\left(\widetilde{\phi}(\varepsilon)-\phi^{\prime}\left(X_{T}^{\omega}\right)\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right] 4.2\right)
\end{aligned}
$$

[^0]To conclude, we have to show that the terms on the right-hand side go to 0 as $\varepsilon \rightarrow 0$. Let us start by considering the first one. We notice that $V_{s}^{\omega+\varepsilon \eta}-V_{s}^{\omega}=\varepsilon \eta_{s}$ which yields

$$
\begin{align*}
\Delta_{\varepsilon \eta} X_{T}^{\omega} & =\int_{t}^{T} \psi\left(s, V_{s}^{\omega+\varepsilon \eta}\right)-\psi\left(s, V_{s}^{\omega}\right) \mathrm{d} B_{s}  \tag{4.3}\\
& =\int_{t}^{T}\left(\left(V_{s}^{\omega+\varepsilon \eta}-V_{s}^{\omega}\right) \int_{0}^{1} \psi^{\prime}\left(s, V_{s}^{\omega}+\lambda\left(V_{s}^{\omega+\varepsilon \eta}-V_{s}^{\omega}\right)\right) \mathrm{d} \lambda\right) \mathrm{d} B_{s}=\varepsilon \int_{t}^{T} \eta_{s} \widetilde{\psi}_{s}(\varepsilon) \mathrm{d} B_{s}
\end{align*}
$$

where $\widetilde{\psi}_{s}(\varepsilon):=\int_{0}^{1} \psi^{\prime}\left(s, V_{s}^{\omega}+\lambda \varepsilon \eta_{s}\right) \mathrm{d} \lambda$. Moreover, for all $\lambda \in(0,1)$, Assumption 2.19 gives

$$
\begin{equation*}
\mathbb{E}_{t, x}\left[\phi^{\prime}\left(\lambda X_{T}^{\omega}+(1-\lambda) X_{T}^{\omega+\varepsilon \eta}\right)^{2}\right] \lesssim 1+\mathbb{E}_{t, x}\left[\left|X_{T}^{\omega}\right|^{2 \kappa_{\phi}}+\left|X_{T}^{\omega+\varepsilon \eta}\right|^{2 \kappa_{\phi}}\right] \tag{4.4}
\end{equation*}
$$

which is bounded by $(2.19)$ and in turn yields the finiteness of the second moment of $\widetilde{\phi}(\varepsilon)$. On the other hand, the local Lipschitz continuity of $\psi^{\prime}(s, \cdot)$ gives

$$
\begin{equation*}
\widetilde{\psi}_{s}(\varepsilon)-\psi^{\prime}\left(s, V_{s}^{\omega}\right)=\int_{0}^{1} \psi^{\prime}\left(s, V_{s}^{\omega}+\lambda \varepsilon \eta_{s}\right)-\psi^{\prime}\left(s, V_{s}^{\omega}\right) \mathrm{d} \lambda \leq \frac{1}{2} \sup _{\varepsilon^{\prime} \leq \varepsilon}\left|\psi^{\prime \prime}\left(s, V_{s}^{\omega}+\varepsilon^{\prime} \eta_{s}\right)\right|\left|\eta_{s}\right| \varepsilon \tag{4.5}
\end{equation*}
$$

For all $p>1$, again by Assumption 2.19 and 2.19 , we have

$$
\begin{aligned}
\mathbb{E}_{t, x}\left[\sup _{\varepsilon^{\prime} \leq \varepsilon}\left|\psi^{\prime \prime}\left(s, V_{s}^{\omega}+\varepsilon^{\prime} \eta_{s}\right)\right|^{p}\right] & \lesssim \mathbb{E}_{t, x}\left[\sup _{\varepsilon^{\prime} \leq \varepsilon}\left(1+\mathrm{e}^{\kappa_{\psi}\left(V_{s}^{\omega}+\varepsilon^{\prime} \eta_{s}\right)}\right)^{p}\right] \lesssim 1+\sup _{\varepsilon^{\prime} \leq \varepsilon} \mathrm{e}^{\kappa_{\psi} p \varepsilon^{\prime} \eta_{s}} \mathbb{E}_{t, x}\left[\mathrm{e}^{\kappa_{\psi} p V_{s}^{\omega}}\right] \\
& \lesssim 1+\mathrm{e}^{\kappa_{\psi} p \varepsilon\|\eta\|_{\mathrm{T}}} \mathbb{E}\left[\mathrm{e}^{\kappa_{\psi} p V_{s}^{\omega}}\right]<\infty .
\end{aligned}
$$

This allows us to show, exploiting (4.3) and Itô's isometry first and then 4.5),

$$
\begin{align*}
\mathbb{E}_{t, x}\left[\left(\frac{\Delta_{\varepsilon \eta} X_{T}^{\omega}}{\varepsilon}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right)^{2}\right] & =\mathbb{E}_{t, x}\left[\left(\int_{t}^{T}\left(\widetilde{\psi}_{s}(\varepsilon)-\psi^{\prime}\left(s, V_{s}^{\omega}\right)\right) \eta_{s} \mathrm{~d} B_{s}\right)^{2}\right] \\
& =\mathbb{E}_{t, x}\left[\int_{t}^{T}\left(\widetilde{\psi}_{s}(\varepsilon)-\psi^{\prime}\left(s, V_{s}^{\omega}\right)\right)^{2} \eta_{s}^{2} \mathrm{~d} s\right] \\
& \leq \sup _{s \in \mathbb{T}} \frac{\mathbb{E}_{t, x}\left[\sup _{\varepsilon^{\prime} \leq \varepsilon}\left|\psi^{\prime \prime}\left(s, V_{s}^{\omega}+\varepsilon^{\prime} \eta_{s}\right)\right|^{2}\right]}{4} \varepsilon^{2} \int_{t}^{T} \eta_{s}^{4} \mathrm{~d} s \tag{4.6}
\end{align*}
$$

which goes to zero as $\varepsilon$ tends to zero. By virtue of Cauchy-Schwarz inequality, the sentence below (4.4) and (4.6), we conclude that the first term of (4.2) tends to zero. For the second term, by definition of $\widetilde{\phi}$, the local Lipschitz continuity of $\phi^{\prime \prime}$, Assumption 2.19 and Equation (2.19), we have

$$
\begin{aligned}
\widetilde{\phi}(\varepsilon)-\phi^{\prime}\left(X_{T}^{\omega}\right) & =\int_{0}^{1}\left(\phi^{\prime}\left(X_{T}^{\omega}+\lambda \Delta_{\varepsilon \eta} X_{T}^{\omega}\right)-\phi^{\prime}\left(X_{T}^{\omega}\right)\right) \mathrm{d} \lambda \\
& \leq \sup _{\alpha \in(0,1)}\left|\phi^{\prime \prime}\left((1-\alpha) X_{T}^{\omega}+\alpha X_{T}^{\omega+\varepsilon \eta}\right)\right| \int_{0}^{1} \lambda\left|\Delta_{\varepsilon \eta} X_{T}^{\omega}\right| \mathrm{d} \lambda \\
& \lesssim\left(1+\left|X_{T}^{\omega}\right|^{\kappa_{\phi}}+\left|X_{T}^{\omega+\varepsilon \eta}\right|^{\kappa_{\phi}}\right)\left|\Delta_{\varepsilon \eta} X_{T}^{\omega}\right|
\end{aligned}
$$

By (4.3), BDG and Hölder's inequality, we have

$$
\mathbb{E}_{t, x}\left[\left(\Delta_{\varepsilon \eta} X_{T}^{\omega}\right)^{4}\right] \lesssim \varepsilon^{4} \int_{t}^{T} \eta_{s}^{4} \mathbb{E}_{t, x}\left[\widetilde{\psi}_{s}(\varepsilon)^{4}\right] \mathrm{d} s
$$

while Assumption 2.19 (ii) yields

$$
\mathbb{E}_{t, x}\left[\psi^{\prime}\left(s, V_{s}^{\omega}+\lambda \varepsilon \eta_{s}\right)^{4}\right] \leq 1+\mathrm{e}^{4 \kappa_{\psi} T} \mathbb{E}_{t, x}\left[\exp \left(4 \kappa_{\psi}\left(V_{s}^{\omega}+\lambda \varepsilon \eta_{s}\right)\right)\right] \leq 1+\mathrm{e}^{4 \kappa_{\psi}\left(\omega_{s}+\lambda \varepsilon \eta_{s}\right)} \mathbb{E}_{t, x}\left[\mathrm{e}^{4 \kappa_{\psi} I_{s}^{t}}\right]
$$

which is uniformly bounded for $s, t \in \mathbb{T}$ since $I_{t}^{s}$ is a Gaussian random variable and, as a consequence, so is $\mathbb{E}_{t, x}\left[\widetilde{\psi}_{s}(\varepsilon)^{4}\right]$. Using Cauchy-Schwarz and $\left(1+\left|X_{T}^{\omega}\right|^{\kappa_{\phi}}+\left|X_{T}^{\omega+\varepsilon \eta}\right|^{\kappa_{\phi}}\right) \in L^{4}$, we conclude that

$$
\lim _{\varepsilon \rightarrow 0} \mathbb{E}_{t, x}\left[\left(\widetilde{\phi}(\varepsilon)-\phi^{\prime}\left(X_{T}^{\omega}\right)\right)^{2}\right]=0
$$

On the other hand, similar computations yield

$$
\sup _{t \in \mathbb{T}} \mathbb{E}_{t, x}\left[\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right)^{2}\right]<\infty
$$

A new application of Cauchy-Schwarz and the last two displays show that the second term of (4.2) tends to zero as well, concluding the proof of the pathwise derivative. The crossed derivative $\partial_{\omega x} u$ can be computed in the same way since $\phi^{\prime \prime}$ is also Lipschitz continuous.

- For the second pathwise derivative, we again consider only the case $\zeta=0$, as the other follow with the same method. First of all, let us recall that we use the same notations as in the previous bullet point with $(t, x)$ fixed. For all $\varepsilon>0, \eta^{(1)}, \eta^{(2)} \in \mathcal{W}_{t}, \omega \in \mathcal{W}$, let us define

$$
\Psi_{t}^{i}(\omega):=\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(i)} \mathrm{d} B_{s}, i=1,2,
$$

so that we are interested in showing that the conditional expectation of the following difference goes to 0 with $\varepsilon$ :

$$
\begin{align*}
& \frac{\phi^{\prime}\left(X_{T}^{\omega+\varepsilon \eta^{(2)}}\right) \Psi_{t}^{1}\left(\omega+\varepsilon \eta^{(2)}\right)-\phi^{\prime}\left(X_{T}^{\omega}\right) \Psi_{t}^{1}(\omega)}{\varepsilon}-\left(\phi^{\prime \prime}\left(X_{T}^{\omega}\right) \Psi_{t}^{1}(\omega) \Psi_{t}^{2}(\omega)+\phi^{\prime}\left(X_{T}^{\omega}\right) \int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}\right) \\
&=\left\{\frac{\phi^{\prime}\left(X_{T}^{\omega+\varepsilon \eta^{(2)}}\right)-\phi^{\prime}\left(X_{T}^{\omega}\right)}{\varepsilon}-\phi^{\prime \prime}\left(X_{T}^{\omega}\right) \Psi_{t}^{2}(\omega)\right\} \Psi_{t}^{1}(\omega)  \tag{4.7}\\
&+\phi^{\prime}\left(X_{T}^{\omega+\varepsilon \eta^{(2)}}\right)\left\{\frac{\Psi_{t}^{1}\left(\omega+\varepsilon \eta^{(2)}\right)-\Psi_{t}^{1}(\omega)}{\varepsilon}-\int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}\right\}  \tag{4.8}\\
&+\int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}\left\{\phi^{\prime}\left(X_{T}^{\omega+\varepsilon \eta^{(2)}}\right)-\phi^{\prime}\left(X_{T}^{\omega}\right)\right\} \tag{4.9}
\end{align*}
$$

The first term, i.e. the one in (4.7), can be dealt with in the same way as 4.2 with $\phi^{\prime}, \psi^{\prime}$ replaced by $\phi^{\prime \prime}, \psi^{\prime \prime}$ and one more application of Cauchy-Schwarz to separate the term in brackets and $\Psi_{t}^{1}(\omega)$. The third term, i.e. the one in (4.9), can be studied similarly to (4.1) exploiting 4.3). We focus our attention on the second term, i.e. the one in 4.8). As in (4.3), we start by writing

$$
\Psi_{t}^{1}\left(\omega+\varepsilon \eta^{(2)}\right)-\Psi_{t}^{1}(\omega)=\int_{t}^{T}\left(\psi^{\prime}\left(s, V_{s}^{\omega+\varepsilon \eta^{(2)}}\right)-\psi^{\prime}\left(s, V_{s}^{\omega}\right)\right) \eta_{s}^{(1)} \mathrm{d} B_{s}=\varepsilon \int_{t}^{T} \widetilde{\psi}_{s}^{\prime}(\varepsilon) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}
$$

where $\widetilde{\psi}_{s}^{\prime}(\varepsilon):=\int_{0}^{1} \psi^{\prime \prime}\left(s, V_{s}^{\omega}+\lambda \varepsilon \eta_{s}^{(2)}\right) \mathrm{d} \lambda$. Therefore,

$$
\begin{align*}
\frac{\Psi_{t}^{1}\left(\omega+\varepsilon \eta^{(2)}\right)-\Psi_{t}^{1}(\omega)}{\varepsilon} & -\int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}=\int_{t}^{T}\left(\widetilde{\psi}_{s}^{\prime}(\varepsilon)-\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}  \tag{4.10}\\
& =\int_{t}^{T}\left(\int_{0}^{1}\left(\psi^{\prime \prime}\left(s, V_{s}^{\omega}+\lambda \varepsilon \eta_{s}^{(2)}\right)-\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)\right) \mathrm{d} \lambda\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}
\end{align*}
$$

Now, as in 4.6), Itô's isometry for (4.10) and the local Lipschitz continuity of $\psi^{\prime \prime}$ yield

$$
\lim _{\varepsilon \rightarrow 0} \mathbb{E}_{t, x}\left[\left(\frac{\Psi_{t}^{1}\left(\omega+\varepsilon \eta^{(2)}\right)-\Psi_{t}^{1}(\omega)}{\varepsilon}-\int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} B_{s}\right)^{2}\right]=0
$$

which concludes the proof.
4.3. Proof of Proposition $\mathbf{2 . 2 3}$. We recall that the pathwise derivatives with respect to a singular direction is the limit of smooth directions, see Equation (2.4). Hence, we introduce the smooth approximation $K^{\delta, t}:=K(\cdot+\delta, t)$ which belongs to $\mathcal{W}_{t}$, such that Proposition 2.22 holds with $\eta=K^{\delta, t}$, and then we pass to the limit $\delta \rightarrow 0$. We show that the conclusions of Proposition 2.22 still hold for the second pathwise derivative, and in the case $\zeta=0$, as the first derivative and the other cases follow in a straightforward way using the same techniques.

- Let us start with the first term of (2.22). We use the identity $a^{2}-b^{2}=(a+b)(a-b)$ and apply Cauchy-Schwarz inequality

$$
\begin{align*}
& \mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) K(s+\delta, t) \mathrm{d} B_{s}\right)^{2}\right]-\mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) K(s, t) \mathrm{d} B_{s}\right)^{2}\right] \\
& \leq \sqrt{2} \mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right)^{2}\left\{\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) K(s+\delta, t) \mathrm{d} B_{s}\right)^{2}+\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right) K(s, t) \mathrm{d} B_{s}\right)^{2}\right\}\right]^{\frac{1}{2}}  \tag{4.11}\\
& \quad \cdot \mathbb{E}_{t, x, \omega}\left[\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right)\{K(s+\delta, t)-K(s, t)\} \mathrm{d} B_{s}\right)^{2}\right]^{\frac{1}{2}}
\end{align*}
$$

Let us start by studying the first term on the right hand side, namely 4.11. For all $p>1$ and $\delta \geq 0$, we apply BDG and Holder's inequalities to get

$$
\begin{align*}
& \mathbb{E}_{t, x, \omega}\left[\left|\int_{t}^{T} K(s+\delta, t) \psi^{\prime}\left(s, V_{s}\right) \mathrm{d} B_{s}\right|^{p}\right] \\
& \leq \mathfrak{b}_{p} \mathbb{E}_{t, x, \omega}\left[\left|\int_{t}^{T} K(s+\delta, t)^{2} \psi^{\prime}\left(s, V_{s}\right)^{2} \mathrm{~d} s\right|^{p / 2}\right] \\
& =\mathfrak{b}_{p} \mathbb{E}_{t, x, \omega}\left[\left|\int_{t}^{T}\left(K(s+\delta, t)^{4 / p} \psi^{\prime}\left(s, V_{s}\right)^{2}\right) K(s+\delta, t)^{\frac{2 p-4}{p}} \mathrm{~d} s\right|^{p / 2}\right] \\
& \leq \mathfrak{b}_{p} \mathbb{E}_{t, x, \omega}\left[\left|\int_{t}^{T}\left(K(s+\delta, t)^{4 / p} \psi^{\prime}\left(s, V_{s}\right)^{2}\right)^{p / 2} \mathrm{~d} r\right| \cdot\left|\int_{t}^{T} K(s+\delta, t)^{\frac{2 p-4}{p} \frac{p}{p-2}} \mathrm{~d} s\right|^{\frac{p}{2} \frac{p-2}{p}}\right] \\
& \leq \mathfrak{b}_{p} \int_{t}^{T} K(s+\delta, t)^{2} \mathbb{E}_{t, x, \omega}\left[\left|\psi^{\prime}\left(s, V_{s}\right)\right|^{p}\right] \mathrm{d} s\left(\int_{t}^{T} K(s+\delta, t)^{2} \mathrm{~d} s\right)^{\frac{p-2}{2}} . \tag{4.12}
\end{align*}
$$

Since $\mathbb{E}_{t, x, \omega}\left[\left|\psi^{\prime}\left(s, V_{s}\right)\right|^{p}\right]=\mathbb{E}\left[\left|\psi^{\prime}\left(s, V_{s}^{t, \omega}\right)\right|^{p}\right]$ is uniformly bounded in $s$, the above is bounded by some constant times $\left(\int_{t}^{T} K(s+\delta, t)^{2} \mathrm{~d} s\right)^{p / 2}$. This holds for any $\delta \geq 0$, hence Hölder's inequality ensures that (4.11) is bounded. Itô's isometry, together with Assumption 2.19 and 2.20 , yields

$$
\begin{aligned}
\mathbb{E}_{t, x, \omega}[ & {\left.\left[\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right)\{K(s+\delta, t)-K(s, t)\} \mathrm{d} B_{s}\right)^{2}\right]=\mathbb{E}\left[\int_{t}^{T} \psi^{\prime}\left(s, V_{s}\right)^{2}\{K(s+\delta, t)-K(s, t)\}^{2} \mathrm{~d} s\right] } \\
& \lesssim \sup _{s \in \mathbb{T}} \mathbb{E}\left[\psi^{\prime}\left(s, V_{s}\right)^{2}\right] \int_{t}^{T}\{K(s+\delta, t)-K(s, t)\}^{2} \mathrm{~d} s \lesssim \delta^{2 H}
\end{aligned}
$$

where the last estimate can be found in [2, Example 2.3] for instance.

- The other term to deal with is, for $\delta>0$,

$$
\mathbb{E}_{t, x, \omega}\left[\phi^{\prime}\left(X_{T}\right) \int_{t}^{T} \psi^{\prime \prime}\left(s, V_{s}\right) K(s+\delta, t)^{2} \mathrm{~d} B_{s}\right]=\mathbb{E}_{t, x, \omega}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}\right) K(s+\delta, t)^{2} \mathrm{~d} s\right]
$$

with the help of Malliavin integration by parts (2.21). Otherwise, when taking $\delta \downarrow 0$, we would obtain the ill-defined integral $\int_{t}^{T} K(s, t)^{2} \mathrm{~d} B_{s}$. Notice that, for $r \leq s, \mathrm{D}_{s} V_{r}=K(s, r)$ and $\overline{\mathrm{D}}_{s} V_{r}=0$. Therefore, $\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right)=\phi^{\prime \prime}\left(X_{T}\right)\left(\mathrm{D}_{s} X_{T}, \overline{\mathrm{D}}_{s} X_{T}\right)^{\top}$ where

$$
\begin{align*}
\overline{\mathrm{D}}_{s} X_{T} & =\bar{\rho} \psi\left(s, V_{s}\right)  \tag{4.13}\\
\mathrm{D}_{s} X_{T} & =\rho \psi\left(s, V_{s}\right)+\int_{s}^{T} \mathrm{D}_{s} \psi\left(r, V_{r}\right) \mathrm{d} B_{r}+\zeta \int_{0}^{T} \mathrm{D}_{s} \psi\left(r, V_{r}\right)^{2} \mathrm{~d} r \\
& =\rho \psi\left(s, V_{s}\right)+\int_{s}^{T} K(r, s) \psi^{\prime}\left(r, V_{r}\right) \mathrm{d} B_{r}+\zeta \int_{s}^{T} K(r, s)\left(\psi^{2}\right)^{\prime}\left(r, V_{r}\right) \mathrm{d} r .
\end{align*}
$$

Since $\psi, \psi^{2},\left(\psi^{2}\right)^{\prime}$ have subexponential growth, the following holds for all $p>1$,

$$
\begin{aligned}
& \mathbb{E}_{t, x, \omega}\left[\left|\psi\left(s, V_{s}\right)\right|^{p}\right]+\mathbb{E}_{t, x, \omega}\left[\left|\int_{s}^{T} K(r, s)\left(\psi^{2}\right)^{\prime}\left(r, V_{r}\right) \mathrm{d} r\right|^{p}\right] \\
& \quad \lesssim 1+\mathrm{e}^{p \kappa_{\psi} T} \sup _{s \in \mathbb{T}} \mathbb{E}_{t, x, \omega}\left[\mathrm{e}^{p \kappa_{\psi} V_{s}}\right]+\mathrm{e}^{p \kappa_{\psi}^{2} T} \sup _{s \in \mathbb{T}} \mathbb{E}_{t, x, \omega}\left[\mathrm{e}^{p \kappa_{\psi}^{2} V_{s}}\right]\left(\int_{s}^{T} K(r, s) \mathrm{d} r\right)^{p}
\end{aligned}
$$

By virtue of (2.19) and $K \in L^{2}(\mathbb{T})$, these terms are uniformly bounded in $L^{p}$ so that the first and the last terms on the righthand side of $\left(4.13\right.$ ) are in $L^{p}$. The same technique as in 4.12 shows the same holds for the second term in the second equation in 4.13); this entails

$$
\begin{equation*}
\sup _{s \in \mathbb{T}} \mathbb{E}\left[\left|\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right), \boldsymbol{\rho}\right\rangle\right|^{p}\right]<\infty \tag{4.14}
\end{equation*}
$$

Hence the proof follows from the estimate

$$
\begin{aligned}
& \left|\mathbb{E}_{t, x, \omega}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}\right)\left\{K(s+\delta, t)^{2}-K(s, t)^{2}\right\} \mathrm{d} s\right]\right| \\
& \leq \sup _{s \in \mathbb{T}} \mathbb{E}_{t, x, \omega}\left[\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}\right)\right]\left|\int_{t}^{T}\left\{K(s+\delta, t)^{2}-K(s, t)^{2}\right\} \mathrm{d} s\right| \lesssim \delta^{2 H}
\end{aligned}
$$

4.4. Proof of Proposition 2.24. As in Definition 2.8, let $(t, x, \omega) \in \bar{\Lambda}$ and $\eta, \eta^{(1)}, \eta^{(2)} \in \mathcal{W}_{t}$ with supports in $[t, t+\delta]$ for some small $\delta>0$. Essentially, the estimates in $x, \omega$ are verified thanks to growth conditions on $\phi, \psi$ and the bounds 2.19 ). The presence of $\eta$ in a Riemann integral (resp. stochastic integral) leads to a bound proportional to $\delta$ (resp. $\sqrt{\delta}$ ). This justifies the estimates of Definition 2.8 with $\alpha=1 / 2$. Finally the continuity of the derivatives is a consequence of the regularity of $\phi$ and $\psi$.
(i) Since $\phi^{\prime}$ and $\phi^{\prime \prime}$ both have polynomial growth, it is clear from 2.19 that $\partial_{x} u$ and $\partial_{x x}^{2} u$ have $G$-growth as introduced in Definition 2.8. Let us note these derivatives do not require the factor $\delta^{\alpha}$ as they are not in the direction of a singular kernel. Turning to the pathwise derivative $\left\langle\partial_{\omega} u, \eta\right\rangle$ given in Proposition 2.22 , we apply Itô's isometry after Hölder's inequality and by the growth of $\phi, \psi$ and 2.19, there exist $G_{1}, G_{2} \in \mathcal{X}$ such that

$$
\begin{aligned}
\mathbb{E}\left[\phi^{\prime}\left(X_{T}^{t, x, \omega}\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s} \mathrm{~d} B_{s}\right] & \leq \mathbb{E}\left[\phi^{\prime}\left(X_{T}^{t, x, \omega}\right)^{2}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_{t}^{t+\delta} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s} \mathrm{~d} B_{s}\right)^{2}\right]^{\frac{1}{2}} \\
& \lesssim\left\|G_{1}(x, \omega)\right\|_{\mathbb{T}} \mathbb{E}\left[\int_{t}^{t+\delta} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right)^{2} \eta_{s}^{2} \mathrm{~d} s\right]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \lesssim\left\|G_{1}(x, \omega)\right\|_{\mathbb{T}} \sup _{s \in \mathbb{T}} \mathbb{E}\left[\psi^{\prime}\left(s, V_{s}^{t, \omega}\right)^{2} \mathrm{~d} s\right]^{\frac{1}{2}}\|\eta\|_{\mathbb{T}} \sqrt{\delta} \\
& \lesssim\left\|G_{1}(x, \omega)\right\|_{\mathbb{T}}\left\|G_{2}(x, \omega)\right\|_{\mathbb{T}}\|\eta\|_{\mathbb{T}} \sqrt{\delta}
\end{aligned}
$$

Here $\|G(x, \omega)\|_{\mathbb{T}}=\sup _{t \in \mathbb{T}}\left|G\left(x, \omega_{t}\right)\right|$. Clearly, $G_{1} G_{2}$ is also in $\mathcal{X}$ by Cauchy-Schwarz inequality, and the second term of $\left\langle\partial_{\omega} u, \eta\right\rangle$ can be dealt with in the same way by applying Jensen's inequality instead of Itô's isometry. This shows the $G$-growth with $\alpha=1 / 2$. The case of $\left\langle\partial_{\omega}\left(\partial_{x} u\right), \eta\right\rangle$ is identical.

For the second pathwise derivative, we must exploit the representation provided in Equation 2.22 . From (4.14) we derive

$$
\int_{t}^{T} \mathbb{E}\left[\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t, x, \omega}\right), \boldsymbol{\rho}\right\rangle\right] \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s \lesssim \int_{t}^{T} \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s \lesssim\left\|\eta_{s}^{(1)}\right\|_{\mathbb{T}}\left\|\eta_{s}^{(2)}\right\|_{\mathbb{T}} \delta
$$

Similar computations show that for some $G \in \mathcal{X}$

$$
\mathbb{E}\left[\zeta \int_{t}^{T}\left(\psi_{s}^{2}\right)^{\prime \prime}\left(V_{s}^{t, \omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s\right] \lesssim\|G(x, \omega)\|_{\mathbb{T}}^{2}\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}} \delta
$$

For the other terms, we apply Hölder's inequality with $\frac{1}{p_{1}}+\frac{1}{p_{2}}+\frac{1}{p_{3}}=1$ to separate the three factors and there exist $G \in \mathcal{X}$ such that

$$
\begin{aligned}
& \mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}^{t, x, \omega}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& \leq\left\|\phi^{\prime \prime}\left(X_{T}^{t, x, \omega}\right)\right\|_{L^{p_{1}}}\left\|\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right\|_{L^{p_{2}}}\left\|\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t, \omega}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right\|_{L^{p_{3}}} \lesssim\|G(x, \omega)\|_{\mathbb{T}}\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}} \delta,
\end{aligned}
$$

where the last inequality follows from BDG inequality and taking the supremum of $\eta^{(1)}, \eta^{(2)}$. The same technique of splitting into three different factors shows that the other terms also yield the same estimates. This concludes Definition 2.8 (i) with $\alpha=1 / 2$.
(ii) For the regularity, we consider $(t, x, \omega),\left(t, x^{\prime}, \omega^{\prime}\right) \in \bar{\Lambda}$. We focus on $\omega$-continuity as $x$-continuity is easier and follows with similar arguments. Hence we fix $(t, x)$, abbreviate $X^{t, x, \omega}$ by $X^{\omega}$ and denote the $L^{2}$ norm under $\mathbb{E}_{t, x}$ as $\|\cdot\|_{L_{t, x}^{2}}$ :

$$
\begin{aligned}
& \mathbb{E}_{t, x}\left[\phi^{\prime}\left(X_{T}^{\omega}\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right]-\mathbb{E}_{t, x}\left[\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s} \mathrm{~d} B_{s}\right] \\
& =\mathbb{E}_{t, x}\left[\left(\phi^{\prime}\left(X_{T}^{\omega}\right)-\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\right) \int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right] \\
& \quad+\mathbb{E}_{t, x}\left[\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s} \mathrm{~d} B_{s}\right)\right] \\
& \leq\left\|\phi^{\prime}\left(X_{T}^{\omega}\right)-\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}}\left\|\int_{t}^{t+\delta} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{s}\right\|_{L_{t, x}^{2}} \\
& \quad+\left\|\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}}\left\|\int_{t}^{t+\delta}\left(\psi^{\prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right) \eta_{s} \mathrm{~d} B_{s}\right\|_{L_{t, x}^{2}}
\end{aligned}
$$

Previous computations showed there exists $G \in \mathcal{X}$ such that

$$
\left\|\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}} \lesssim\left\|G\left(x, \omega^{\prime}\right)\right\|_{\mathbb{T}}, \quad \text { and } \quad\left\|\int_{t}^{t+\delta} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s} \mathrm{~d} B_{S}\right\|_{L_{t, x}^{2}} \lesssim\|G(x, \omega)\|_{\mathbb{T}}\|\eta\|_{\mathbb{T}} \sqrt{\delta}
$$

Similarly to 4.3 we have

$$
\left\|\int_{t}^{t+\delta}\left(\psi^{\prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right) \eta_{s} \mathrm{~d} B_{s}\right\|_{L_{t, x}^{2}} \lesssim\|G(x, \omega)\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\|\eta\|_{\mathbb{T}} \sqrt{\delta}
$$

Starting with a decomposition similar to the one in 4.3), then applying Itô's isometry and the fact that $\eta$ is uniformly bounded over time together with Assumption 2.19, we obtain

$$
\begin{aligned}
\left\|\int_{t}^{t+\delta}\left(\psi^{\prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right) \eta_{s} \mathrm{~d} B_{s}\right\|_{L_{t, x}^{2}} & =\left\|\int_{t}^{t+\delta} \eta_{s}\left(V_{s}^{\omega}-V_{s}^{\omega^{\prime}}\right) \int_{0}^{1} \psi^{\prime \prime}\left(s, \lambda V_{s}^{\omega}+(1-\lambda) V_{s}^{\omega^{\prime}}\right) \mathrm{d} \lambda \mathrm{~d} B_{s}\right\|_{L_{t, x}^{2}} \\
& =\mathbb{E}_{t, x}\left[\int_{t}^{t+\delta} \eta_{s}^{2}\left(\omega_{s}-\omega_{s}^{\prime}\right)^{2}\left(\int_{0}^{1} \psi^{\prime \prime}\left(s, \lambda V_{s}^{\omega}+(1-\lambda) V_{s}^{\omega^{\prime}}\right) \mathrm{d} \lambda\right)^{2} \mathrm{~d} s\right]^{1 / 2} \\
& \lesssim\|\eta\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}} \mathbb{E}\left[\int _ { t } ^ { t + \delta } \int _ { 0 } ^ { 1 } \left(1+\mathrm{e}^{\left.\left.\kappa_{\psi}\left(\lambda V_{s}^{\omega}+(1-\lambda) V_{s}^{\omega^{\prime}}\right)\right)^{2} \mathrm{~d} \lambda \mathrm{~d} s\right]^{1 / 2}}\right.\right. \\
& \lesssim\|\eta\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\left(1+\int_{t}^{t+\delta} \mathbb{E}\left[\mathrm{e}^{2 \kappa_{\psi} I_{s}^{t}}\right] \int_{0}^{1} \mathrm{e}^{2 \kappa_{\psi}\left(\lambda \omega_{s}+(1-\lambda) \omega_{s}^{\prime}\right)} \mathrm{d} \lambda \mathrm{~d} s\right)^{1 / 2} \\
& \lesssim\|\eta\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\left(1+\int_{t}^{t+\delta}\left(\mathrm{e}^{2 \kappa_{\psi} \omega_{s}}+\mathrm{e}^{2 \kappa_{\psi} \omega_{s}^{\prime}}\right) \mathrm{d} s\right)^{1 / 2} \\
& \lesssim\left(\|G(x, \omega)\|_{\mathbb{T}}+\left\|G\left(x, \omega^{\prime}\right)\right\|_{\mathbb{T}}\right)\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\|\eta\|_{\mathbb{T}} \sqrt{\delta}
\end{aligned}
$$

where we used the convexity of the exponential. Following the idea of (4.1), we also know by local Lipschitz continuity and Cauchy-Schwarz inequality

$$
\begin{aligned}
\left\|\phi^{\prime}\left(X_{T}^{\omega}\right)-\phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}} & \leq\left\|X_{T}^{\omega}-X_{T}^{\omega^{\prime}}\right\|_{L_{t, x}^{4}}\left\|\int_{0}^{1} \phi^{\prime \prime}\left(\lambda X_{T}^{\omega}+(1-\lambda) X_{T}^{\omega^{\prime}}\right) \mathrm{d} \lambda\right\|_{L_{t, x}^{4}} \\
& \lesssim\left(\|G(x, \omega)\|_{\mathbb{T}}+\left\|G\left(x, \omega^{\prime}\right)\right\|_{\mathbb{T}}\right)\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}
\end{aligned}
$$

where, to get the bound for the integral, we have exploited the fact that, by Assumption 2.19, monotonicity and an application of Lemma 2.21, we have

$$
\begin{aligned}
\left\|\int_{0}^{1} \phi^{\prime \prime}\left(\lambda X_{T}^{\omega}+(1-\lambda) X_{T}^{\omega^{\prime}}\right) \mathrm{d} \lambda\right\|_{L_{t, x}^{4}} & \lesssim\left\|\int_{0}^{1}\left(1+\left(\lambda X_{T}^{\omega}+(1-\lambda) X_{T}^{\omega^{\prime}}\right)^{\kappa_{\phi}}\right) \mathrm{d} \lambda\right\|_{L_{t, x}^{4}} \\
& \lesssim 1+\left\|\int_{0}^{1}\left(\lambda X_{T}^{\omega}\right)^{\kappa_{\phi}}+\left((1-\lambda) X_{T}^{\omega^{\prime}}\right)^{\kappa_{\phi}} \mathrm{d} \lambda\right\|_{L_{t, x}^{4}} \\
& \lesssim 1+\left\|\left(X_{T}^{\omega}\right)^{\kappa_{\phi}}\right\|_{L_{t, x}^{4}}+\left\|\left(X_{T}^{\omega^{\prime}}\right)^{\kappa_{\phi}}\right\|_{L_{t, x}^{4}} \\
& \lesssim 1+\left(|x|^{4 \kappa_{\phi}}+\mathrm{e}^{2 \kappa_{\psi} 4 \kappa_{\psi}\|\omega\|_{T}}\right)^{\frac{1}{4}}+\left(|x|^{4 \kappa_{\phi}}+\mathrm{e}^{2 \kappa_{\psi} 4 \kappa_{\psi}}\left\|\omega^{\prime}\right\|_{T}\right)^{\frac{1}{4}} \\
& \lesssim 1+|x|^{4 \kappa_{\phi}}+\mathrm{e}^{2 \kappa_{\psi} 4 \kappa_{\psi}\|\omega\|_{T}}+\mathrm{e}^{2 \kappa_{\psi} 4 \kappa_{\psi}\left\|\omega^{\prime}\right\|_{T}}
\end{aligned}
$$

which corresponds to $\|G(x, \omega)\|_{\mathbb{T}}+\left\|G\left(x, \omega^{\prime}\right)\right\|_{\mathbb{T}}$ in the bound above. Once again, the other term follows from the same steps.

Let us move on to the second derivative with representation 2.22 . The first term in the representation can be handed in the following way. Let us notice once more that without loss of generality we restrict our study to the case $\zeta=0$, as the additional terms are studied with similar computations.

$$
\begin{aligned}
& \mathbb{E}_{t, x}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s\right]-\mathbb{E}_{t, x}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s\right] \\
& =\mathbb{E}_{t, x}\left[\int _ { t } ^ { t + \delta } \left\{\left(\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega}\right), \boldsymbol{\rho}\right\rangle-\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle\right) \psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)\right.\right. \\
& \\
& \left.\left.\quad+\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle\left(\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime \prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right)\right\} \eta_{s}^{(1)} \eta_{s}^{(2)} \mathrm{d} s\right] \\
& \leq \int_{t}^{t+\delta}\left\|\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega}\right), \boldsymbol{\rho}\right\rangle-\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle\right\|_{L_{t, x}^{2}}\left\|\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)\right\|_{L_{t, x}^{2}} \mathrm{~d} s\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}} \\
& \quad+\int_{t}^{t+\delta}\left\|\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle\right\|_{L_{t, x}^{2}}\left\|\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime \prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}} \mathrm{~d} s\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}}
\end{aligned}
$$

Concerning the second term on the right-hand side, we have already seen that $\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle$ is bounded in $L^{2}$ and, since $\psi^{\prime \prime}$ is $C^{1},\left\|\psi^{\prime \prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime \prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right\|_{L^{2}} \lesssim\|G(x, \omega)\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}$. On the other hand, for the first term, we see from $(4.13)$ that

$$
\begin{aligned}
& \left\|\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega}\right), \boldsymbol{\rho}\right\rangle-\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{\omega^{\prime}}\right), \boldsymbol{\rho}\right\rangle\right\|_{L_{t, x}^{2}} \\
& =\left\|\rho \phi^{\prime \prime}\left(X_{T}^{\omega}\right) \mathrm{D}_{s} X_{T}^{\omega}+\bar{\rho} \phi^{\prime \prime}\left(X_{T}^{\omega}\right) \overline{\mathrm{D}}_{s} X_{T}^{\omega}-\rho \phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right) \mathrm{D}_{s} X_{T}^{\omega^{\prime}}+\bar{\rho} \phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right) \overline{\mathrm{D}}_{s} X_{T}^{\omega^{\prime}}\right\|_{L_{t, x}^{2}}
\end{aligned}
$$

and, focusing on the first term, Cauchy-Schwarz inequality yields

$$
\begin{aligned}
& \left\|\phi^{\prime \prime}\left(X_{T}^{\omega}\right) \mathrm{D}_{s} X_{T}^{\omega}-\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right) \mathrm{D}_{s} X_{T}^{\omega^{\prime}}\right\|_{L_{t, x}^{2}} \\
& \leq\left\|\left(\phi^{\prime \prime}\left(X_{T}^{\omega}\right)-\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\right) \mathrm{D}_{s} X_{T}^{\omega}\right\|_{L_{t, x}^{2}}+\left\|\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\left(\mathrm{D}_{s} X_{T}^{\omega}-\mathrm{D}_{s} X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{2}} \\
& \lesssim\left\|\phi^{\prime \prime}\left(X_{T}^{\omega}\right)-\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{4}}\left\|\mathrm{D}_{s} X_{T}^{\omega}\right\|_{L_{t, x}^{4}}+\left\|\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{4}}\left(\left\|\psi\left(s, V_{s}^{\omega}\right)-\psi\left(s, V_{s}^{\omega^{\prime}}\right)\right\|_{L_{t, x}^{4}}\right. \\
& \left.\quad+\left\|\int_{s}^{T} K(r, s)\left(\psi_{r}^{\prime}\left(V_{r}^{\omega}\right)-\psi_{r}^{\prime}\left(V_{r}^{\omega^{\prime}}\right)\right) \mathrm{d} B_{r}\right\|_{L_{t, x}^{4}}+\left\|\int_{s}^{T} K(r, s)\left(\left(\psi_{r}^{2}\right)^{\prime}\left(V_{r}^{\omega}\right)-\left(\psi_{r}^{2}\right)^{\prime}\left(V_{r}^{\omega^{\prime}}\right)\right) \mathrm{d} r\right\|_{L_{t, x}^{4}}\right)
\end{aligned}
$$

The regularity of $\psi, \psi^{\prime}, \psi^{\prime \prime}$ allow us to conclude, using similar computations as before, that this term is smaller than $\|G(x, \omega)\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}} \delta$.

For the next term, we separate this triple in the following way

$$
\begin{aligned}
& \mathbb{E}_{t, x}\left[\phi^{\prime \prime}\left(X_{T}^{\omega}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& -\mathbb{E}_{t, x}\left[\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& =\mathbb{E}_{t, x}\left[\left(\phi^{\prime \prime}\left(X_{T}^{\omega}\right)-\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& \quad+\mathbb{E}_{t, x}\left[\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\left(\int_{t}^{T}\left(\psi^{\prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right]
\end{aligned}
$$

$$
+\mathbb{E}_{t, x}\left[\phi^{\prime \prime}\left(X_{T}^{\omega^{\prime}}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T}\left(\psi^{\prime}\left(s, V_{s}^{\omega}\right)-\psi^{\prime}\left(s, V_{s}^{\omega^{\prime}}\right)\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right]
$$

and then applying Hölder to each of them, the same arguments as in point (i) show that is bounded by $\|G(x, \omega)\|_{\mathbb{T}}\left\|\omega-\omega^{\prime}\right\|_{\mathbb{T}}\left\|\eta^{(1)}\right\|_{\mathbb{T}}\left\|\eta^{(2)}\right\|_{\mathbb{T}} \delta$.
(iii) We are left with checking the last condition, that is continuity of the pathwise derivatives on $\bar{\Lambda}$. We just showed they were (uniformly) continuous in $\omega$ thus time continuity is the last one remaining. We only show it for the second derivative, using once again the representation 2.22 . We fix $x, \omega$; as in point (ii) of this proof,

$$
\begin{align*}
& \mathbb{E}_{x, \omega}\left[\phi^{\prime \prime}\left(X_{T}^{t+\delta}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& -\mathbb{E}_{x, \omega}\left[\phi^{\prime \prime}\left(X_{T}^{t}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \\
& =\mathbb{E}_{x, \omega}\left[\left(\phi^{\prime \prime}\left(X_{T}^{t+\delta}\right)-\phi^{\prime \prime}\left(X_{T}^{t}\right)\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right]  \tag{4.15}\\
& +\mathbb{E}_{x, \omega}\left[\phi^{\prime \prime}\left(X_{T}^{t}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right]  \tag{4.16}\\
& +\mathbb{E}_{x, \omega}\left[\phi^{\prime \prime}\left(X_{T}^{t}\right)\left(\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}\right)\left(\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(2)} \mathrm{d} B_{s}\right)\right] \tag{4.17}
\end{align*}
$$

For the first line 4.15, we apply Cauchy-Schwarz inequality to focus on the difference

$$
\left\|\phi^{\prime \prime}\left(X_{T}^{t+\delta}\right)-\phi^{\prime \prime}\left(X_{T}^{t}\right)\right\|_{L_{x, \omega}^{2}} \leq\left\|X_{T}^{t+\delta}-X_{T}^{t}\right\|_{L_{x, \omega}^{4}}\left\|\int_{0}^{1} \phi^{\prime \prime \prime}\left(\lambda X_{T}^{t+\delta}+(1-\lambda) X_{T}^{t}\right) \mathrm{d} \lambda\right\|_{L_{x, \omega}^{4}}
$$

Furthermore, by BDG and Jensen inequalities

$$
\begin{aligned}
\left\|X_{T}^{t+\delta}-X_{T}^{t}\right\|_{L_{x, \omega}^{4}}^{4} & =\mathbb{E}_{x, \omega}\left[\left(\int_{t+\delta}^{T}\left(\psi\left(s, V_{s}^{t+\delta}\right)-\psi\left(s, V_{s}^{t}\right)\right) \mathrm{d} B_{s}+\int_{t}^{t+\delta} \psi\left(s, V_{s}^{t}\right) \mathrm{d} B_{s}\right)^{4}\right] \\
& \leq 8 \mathfrak{b}_{4} T \int_{t}^{T} \mathbb{E}_{x, \omega}\left[\left|\psi\left(s, V_{s}^{t+\delta}\right)-\psi\left(s, V_{s}^{t}\right)\right|^{4}\right] \mathrm{d} s+8 \delta \mathfrak{b}_{4} \int_{t}^{t+\delta} \mathbb{E}_{x, \omega}\left[\psi\left(s, V_{s}\right)^{4}\right] \mathrm{d} s
\end{aligned}
$$

where the second term clearly goes to zero as $\delta$ tends to zero. For the first one, we write

$$
\begin{equation*}
\mathbb{E}_{x, \omega}\left[\left|\psi\left(s, V_{s}^{t+\delta}\right)-\psi\left(s, V_{s}^{t}\right)\right|^{4}\right]=\mathbb{E}_{x, \omega}\left[\left|\left(V_{s}^{t+\delta}-V_{s}^{t}\right) \int_{0}^{1} \psi^{\prime}\left(s, V_{s}^{t}+\lambda\left(V_{s}^{t+\delta}-V_{s}^{t}\right)\right) \mathrm{d} \lambda\right|^{4}\right] \tag{4.18}
\end{equation*}
$$

where the integral is bounded in $L^{8}$ as in (2.19) and

$$
\mathbb{E}_{x, \omega}\left[\left|V_{s}^{t+\delta}-V_{s}^{t}\right|^{8}\right]=\mathbb{E}_{x, \omega}\left[\left|\int_{t}^{t+\delta} K(s, r) \mathrm{d} W_{r}\right|^{8}\right] \leq \mathfrak{b}_{8}\left(\int_{t}^{t+\delta} K(s, r)^{2} \mathrm{~d} r\right)^{4} \leq \mathfrak{b}_{8} \delta^{8 H}
$$

Yet another application of Cauchy-Schwarz yields that 4.15 goes to zero with $\delta$.
The second and third lines $(4.16)$ and $(4.17)$ are identical hence we only deal with the second one. We decompose further
$\int_{t+\delta}^{T} \psi^{\prime}\left(s, V_{s}^{t+\delta}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}-\int_{t}^{T} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}=\int_{t+\delta}^{T}\left(\psi^{\prime}\left(s, V_{s}^{t+\delta}\right)-\psi^{\prime}\left(s, V_{s}^{t}\right)\right) \eta_{s}^{(1)} \mathrm{d} B_{s}+\int_{t}^{t+\delta} \psi^{\prime}\left(s, V_{s}^{t}\right) \eta_{s}^{(1)} \mathrm{d} B_{s}$

The integral between $t$ and $t+\delta$ was already studied point (ii) and we showed these tend to zero in $L^{p}$ norm as $\delta \rightarrow 0$. Since $\psi^{\prime}$ is also $C^{1}$, the same computations as above show that $\mathbb{E}_{x, \omega}\left[\left|\psi^{\prime}\left(s, V_{s}^{t+\delta}\right)-\psi^{\prime}\left(s, V_{s}^{t}\right)\right|^{4}\right]$ goes to zero, which concludes the time continuity of the first term of the representation in 2.22 .

Regarding the next term,

$$
\begin{aligned}
& \mathbb{E}_{x, \omega}\left[\int_{t+\delta}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t+\delta}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t+\delta}\right) K(s, t+\delta)^{2} \mathrm{~d} s\right]-\mathbb{E}_{x, \omega}\left[\int_{t}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right) K(s, t)^{2} \mathrm{~d} s\right] \\
& =\mathbb{E}_{x, \omega}\left[\int_{t+\delta}^{T}\left\{\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t+\delta}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t+\delta}\right) K(s, t+\delta)^{2}-\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right) K(s, t)^{2}\right\} \mathrm{d} s\right. \\
& \left.\quad+\int_{t}^{t+\delta}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right) K(s, t)^{2} \mathrm{~d} s\right] \\
& =\mathbb{E}_{x, \omega}\left[\int_{t+\delta}^{T}\left\{\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t+\delta}\right), \boldsymbol{\rho}\right\rangle-\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle\right\} \psi^{\prime \prime}\left(s, V_{s}^{t+\delta}\right) K(s, t+\delta)^{2} \mathrm{~d} s\right] \\
& \quad+\mathbb{E}_{x, \omega}\left[\int_{t+\delta}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle\left\{\psi^{\prime \prime}\left(s, V_{s}^{t+\delta}\right)-\psi^{\prime \prime}\left(s, V_{s}^{t}\right)\right\} K(s, t+\delta)^{2} \mathrm{~d} s\right] \\
& \quad+\mathbb{E}_{x, \omega}\left[\int_{t+\delta}^{T}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right)\left\{K(s, t+\delta)^{2}-K(s, t)^{2}\right\} \mathrm{d} s\right] \\
& \quad+\mathbb{E}_{x, \omega}\left[\int_{t}^{t+\delta}\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right) K(s, t)^{2} \mathrm{~d} s\right] .
\end{aligned}
$$

Using the expression 4.13) and similar techniques as before, the first term boils down to evaluating differences of the type (4.18), albeit with $\psi^{\prime}$ instead of $\psi$, which we proved tends to zero as $\delta \rightarrow 0$. The second term follows identically with $\psi^{\prime \prime}$ this time. For the third one we apply Cauchy-Schwarz inequality to isolate the term

$$
\left|\int_{t+\delta}^{T}\left\{K(s, t+\delta)^{2}-K(s, t)^{2}\right\} \mathrm{d} s\right| \lesssim \delta^{2 H}
$$

Finally, the last term is smaller in absolute value than

$$
\sup _{s \in[t, t+\delta]} \mathbb{E}_{x, \omega}\left[\left\langle\mathbf{D}_{s} \phi^{\prime}\left(X_{T}^{t}\right), \boldsymbol{\rho}\right\rangle \psi^{\prime \prime}\left(s, V_{s}^{t}\right)\right] \int_{t}^{t+\delta} K(s, t)^{2} \mathrm{~d} s \lesssim \delta^{2 H}
$$

The other terms of 2.22 can be shown to be continuous using the same techniques. This concludes Definition 2.8 (iii) and hence the proof of the Proposition.

## 5. Proof of Proposition 3.10 - the $\mathfrak{D}$ terms

In Section 5.1 we introduce and describe set notations crucial to the proofs. Then, we develop the different cases of Proposition 3.10 separately:

- Computations for $\left|\mathfrak{D}_{1}^{n 1}\left(s_{n}\right)\right|$ and $\left|\mathfrak{D}_{2}^{n 1}\left(s_{n}\right)\right|$ in Section 5.2.
- Computations for $\left|\mathfrak{D}_{1}^{n 2}\left(s_{n}\right)\right|$ in Section 5.3
- Computations for $\left|\mathfrak{D}_{2}^{n 2}\left(s_{n}\right)\right|$ in Section 5.4
5.1. Notations for Malliavin derivatives in the chaos expansion. To streamline the notations in the proofs, given a set of integers $\boldsymbol{N} \subset \mathbb{N}$ and $\boldsymbol{s}_{\boldsymbol{N}}:=\left(s_{i}\right)_{i \in \boldsymbol{N}}$, we introduce

$$
\begin{align*}
\mathbf{K}\left(t, s_{N}\right) & :=\prod_{i \in \boldsymbol{N}} K\left(t, s_{i}\right)  \tag{5.1}\\
s_{\llbracket k_{1}, k_{2} \rrbracket} & :=\left(s_{k_{1}}, \ldots, s_{k_{2}}\right), \quad 1 \leq k_{1} \leq k_{2} \leq n, \quad s_{k}:=s_{\llbracket 1, k \rrbracket}
\end{align*}
$$

Furthermore, we may write $\mathrm{D}_{\boldsymbol{s}_{n}}$ instead of $\mathrm{D}_{\boldsymbol{s}_{n}}^{n}$ whenever the dimension $n$ is clear. We establish a set of notations, crucial to express the chain rule for $X \in \mathbb{D}^{n, 2}$ and $f \in C^{n}$ :

$$
\begin{equation*}
\mathrm{D}^{n} f(X)=\sum_{k=1}^{n} f^{(k)}(X) \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}} \mathcal{D} X \tag{5.2}
\end{equation*}
$$

Let $\mathcal{U}$ be a non-empty set of natural numbers. For $1 \leq k \leq|\mathcal{U}|=: n$, set $\mathcal{I}_{k}^{\mathcal{U}}$ to be the set of sets of ordered vectors obtained from all the partitions in $k$ non-empty subsets of the set $\mathcal{U}$. The cardinality of $\mathcal{I}_{k}^{\mathcal{U}}$ is $S(n, k)=\{n, k\}=\frac{1}{k!} \sum_{i=0}^{k}(-1)^{i}\binom{k}{i}(k-i)^{n}$, that is the notation for the so-called "Stirling number of the second kind" (or Sirling partition number), which is indeed the number of partitions in $k$ non-empty subsets of a set of $n$ elements. For $j \in \llbracket S(n, k) \rrbracket$, denote $\mathcal{I}_{k}^{\mathcal{U}}(j)$ the $j^{\text {th }}$ element in this collection, which is a vector.

For clarity, when $\mathcal{U}=\llbracket n \rrbracket$, we write $\mathcal{U}^{c}$ its complement in $\llbracket n \rrbracket$ and $\mathcal{I}_{k}^{n}$ in place of $\mathcal{I}_{k}^{\llbracket n \rrbracket}$ so that, for example, for $n=1$ (resp. 2 and 3 ) and $k=1$ (resp. $k \in\{1,2\}$ and $k \in\{1,2,3\}$ ), we have

$$
\begin{aligned}
\mathcal{I}_{1}^{1} & =\left\{\mathcal{I}_{1}^{1}(1)\right\}=\left\{\left\{\mathcal{I}_{1}^{1}(1)_{1}\right\}\right\}=\{\{(1)\}\} \\
\mathcal{I}_{1}^{2} & =\left\{\mathcal{I}_{1}^{2}(1)\right\}=\left\{\left\{\mathcal{I}_{1}^{2}(1)_{1}\right\}\right\}=\{\{(1,2)\}\} \\
\mathcal{I}_{2}^{2} & =\left\{\mathcal{I}_{2}^{2}(1)\right\}=\left\{\left\{\mathcal{I}_{2}^{2}(1)_{1}, \mathcal{I}_{2}^{2}(1)_{2}\right\}\right\}=\{\{(1),(2)\}\} \\
\mathcal{I}_{1}^{3} & =\left\{\mathcal{I}_{1}^{3}(1)\right\}=\left\{\left\{\mathcal{I}_{1}^{3}(1)_{1}\right\}\right\}=\{\{(1,2,3)\}\} \\
\mathcal{I}_{2}^{3} & =\left\{\mathcal{I}_{2}^{3}(1), \mathcal{I}_{2}^{3}(2), \mathcal{I}_{2}^{3}(3)\right\}=\left\{\left\{\mathcal{I}_{2}^{3}(1)_{1}, \mathcal{I}_{2}^{3}(1)_{2}\right\},\left\{\mathcal{I}_{2}^{3}(2)_{1}, \mathcal{I}_{2}^{3}(2)_{2}\right\},\left\{\mathcal{I}_{2}^{3}(3)_{1}, \mathcal{I}_{2}^{3}(3)_{2}\right\}\right\} \\
& =\{\{(1),(2,3)\},\{(2),(1,3)\},\{(3),(1,2)\}\} \\
\mathcal{I}_{3}^{3} & =\left\{\mathcal{I}_{3}^{3}(1)\right\}=\left\{\left\{\mathcal{I}_{3}^{3}(1)_{1}, \mathcal{I}_{3}^{3}(1)_{2}, \mathcal{I}_{3}^{3}(1)_{3}\right\}\right\}=\{\{(1),(2),(3)\}\}
\end{aligned}
$$

Let $P(\llbracket n \rrbracket)$ be the power set of $\llbracket n \rrbracket$ and define the function ${ }^{2} \mathfrak{o}: P(\llbracket n \rrbracket) \rightarrow \bigcup_{k=1}^{n} \bigcup_{j=1}^{S(n, k)} \mathcal{I}_{k}^{n}(j)$ mapping any set of natural numbers into an ordered vector whose components are the elements of the set itself, e.g. $\mathfrak{o}(\{7,2,87,33,60\})=(2,7,33,60,87)$. For $0 \leq s_{1} \leq \cdots \leq s_{n} \leq t \leq T$, set $s_{n}:=\left(s_{1}, \ldots, s_{n}\right) \in \mathbb{T}^{n}$, and define $\mathcal{D}_{k}^{\mathcal{U}}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}\right)$ as the collection of Malliavin differential operators given by

$$
\begin{equation*}
\mathcal{D}_{k}^{\mathcal{U}}\left(s_{\mathfrak{o}(\mathcal{U})}\right):=\left\{\mathcal{D}: \mathcal{D}=\prod_{l=1}^{k} \mathrm{D}_{\boldsymbol{s}_{\mathcal{T}_{k}^{u}(j) l}}, j \in \llbracket 1, S(n, k) \rrbracket\right\}, \tag{5.3}
\end{equation*}
$$

where $\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}=\left(s_{i}\right)_{i \in \mathfrak{o}(\mathcal{U})}$. When $\mathcal{U}=\llbracket n \rrbracket$, this reduces to

$$
\begin{equation*}
\mathcal{D}_{k}^{n}\left(s_{n}\right)=\mathcal{D}_{k}^{\llbracket n \rrbracket}\left(s_{n}\right)=\left\{\mathcal{D}: \mathcal{D}=\prod_{l=1}^{k} \mathrm{D}_{\boldsymbol{s}_{k}^{n}(j)_{l}}, j \in \llbracket 1, S(n, k) \rrbracket\right\} \tag{5.4}
\end{equation*}
$$

For example, for $n=1$ (resp. 2 and 3 ) and $k=1$ (resp. $k \in\{1,2\}$ and $k \in\{1,2,3\}$ ), we have

$$
\begin{aligned}
& \mathcal{D}_{1}^{1}\left(s_{1}\right)=\left\{\mathrm{D}_{s_{1}}\right\} ; \\
& \mathcal{D}_{1}^{2}\left(s_{2}\right)=\left\{\mathrm{D}_{s_{1} s_{2}}\right\}, \quad \mathcal{D}_{2}^{2}\left(s_{2}\right)=\left\{\mathrm{D}_{s_{1}}, \mathrm{D}_{s_{2}}\right\} \\
& \mathcal{D}_{1}^{3}\left(s_{3}\right)=\left\{\mathrm{D}_{s_{1} s_{2} s_{3}}\right\}, \quad \mathcal{D}_{2}^{3}\left(s_{3}\right)=\left\{\mathrm{D}_{s_{1}} \mathrm{D}_{s_{2} s_{3}}, \mathrm{D}_{s_{2}} \mathrm{D}_{s_{1} s_{3}}, \mathrm{D}_{s_{3}} \mathrm{D}_{s_{1} s_{2}}\right\}, \quad \mathcal{D}_{3}^{3}\left(s_{3}\right)=\left\{\mathrm{D}_{s_{1}} \mathrm{D}_{s_{2}} \mathrm{D}_{s_{3}}\right\} .
\end{aligned}
$$

Exploiting further the natural structure of $k$-partitions, we can rewrite any differential operator $\mathcal{D} \in$ $\mathcal{D}_{k}^{\mathcal{U}}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}\right)$ in a coincise yet clear way. Fix $k \in \llbracket n \rrbracket$ and let $\mathcal{D}=\mathcal{D}_{k}^{\mathcal{U}}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}\right)(j)$, for some $j \in \llbracket 1, S(n, k) \rrbracket$. Notice that the definition in (5.3) (resp. (5.4)) establishes a one-to-one correspondence between a triple of the form $(\mathcal{U}, k, j)$ (resp. $(n, k, j))$ and the corresponding Malliavin operator $\mathcal{D}$, so that in the following we will often write $\mathcal{I}_{\mathcal{D}}$ in place of $\mathcal{I}_{k}^{\mathcal{U}}(j)$ (resp. $\left.\mathcal{I}_{k}^{n}(j)\right)$. Without loss of generality we assume that the vectors in the set are ordered in the following way, and subscripts and superscripts $\mathcal{U}, k$ and $j$ are omitted here to make the notation less cluttered:

$$
\mathcal{I}_{\mathcal{D}}^{1}:=\mathcal{I}_{k}^{\mathcal{U}}(j)_{1}=\left(j_{1}^{1}, \ldots, j_{n_{1}}^{1}\right), \quad \ldots \ldots \quad \mathcal{I}_{\mathcal{D}}^{k}:=\mathcal{I}_{k}^{\mathcal{U}}(j)_{k}=\left(j_{1}^{k}, \ldots, j_{n_{k}}^{k}\right),
$$

[^1]with $j_{n_{1}}^{1}<\cdots<j_{n_{k}}^{k}$ and $j_{1}^{l}<\cdots<j_{n_{l}}^{l}$, for $l \in \llbracket 1, k \rrbracket$. We set $\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}:=\left(j_{1}^{l}, \ldots, j_{n_{l}-1}^{l}\right) \sim \mathcal{I}_{\mathcal{D}}^{l}-\left(\mathbf{0}, j_{n_{l}}^{l}\right)=$ $\left(j_{1}^{l}, \ldots, j_{n_{l}-1}^{l}, 0\right)$. Then this specific $\mathcal{D}$ can be rewritten as
$$
\mathcal{D}=\prod_{l=1}^{k} \mathrm{D}_{s_{\mathcal{X}_{\mathcal{D}}^{l}}}, \quad \text { where } \quad \mathrm{D}_{\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}}:=\mathrm{D}_{s_{s_{1}}, \ldots, \bar{s}_{l}}, \quad \text { and } \bar{s}_{l}:=s_{j_{n_{l}}^{l}} .
$$

Example 5.1. With $\mathcal{U}=\llbracket 1,3 \rrbracket$ and $\mathcal{I}_{2}^{3}=\{\{(1),(2,3)\},\{(2),(1,3)\},\{(3),(1,2)\}\}$ given above ( $n=3$, $k=2)$, then $S(n, k)=3$ and

$$
\begin{array}{ll}
j=1: & \mathcal{I}_{2}^{\mathcal{U}}(1)=\{(1),(2,3)\}, \\
j=2: & n_{1}=1, n_{2}=2, \quad \text { and } \\
\mathcal{I}_{2}^{u}(2)=\{(2),(1,3)\}, & n_{1}=1, n_{2}=2, \quad \text { and }\left\{\begin{array}{l}
\mathcal{I}^{1}=\mathcal{I}_{2}^{u}(1)_{1}=(1), \\
\mathcal{I}^{2}=\mathcal{I}_{2}^{u}(1)_{2}=(2,3), \\
\mathcal{I}^{1}=\mathcal{I}_{2}^{u}(2)_{1}=(2), \\
\mathcal{I}^{2}=\mathcal{I}_{2}^{u}(2)_{2}=(1,3), \\
\mathcal{I}^{1}=\mathcal{I}_{2}^{u}(3)_{1}=(1,2), \\
\mathcal{I}^{2}=\mathcal{I}_{2}^{u}(3)_{2}=(3) .
\end{array}\right.
\end{array}
$$

These notations give meaning to the chain rule formula (5.2). For completeness, for any $\mathcal{D} \in \mathcal{D}_{k}^{n}$ and $l \in \llbracket 1, k \rrbracket$, we can express

$$
\begin{align*}
\mathrm{D}_{\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}} X_{T}^{t, \bar{X}_{t}} & =\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{r}\right) \prod_{k \in \mathcal{I}_{\mathcal{D}}^{l}} K\left(r, s_{k}\right) \mathrm{d} B_{r}+\rho \psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right) \prod_{k \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} K\left(\bar{s}_{l}, s_{k}\right) \\
& =\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{r}\right) \mathbf{K}\left(r, s_{\mathcal{I}_{\mathfrak{D}}^{l}}\right) \mathrm{d} B_{r}+\rho \psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathfrak{D}}}\right), \tag{5.5}
\end{align*}
$$

using the vector notations in Equation (5.1), and where only the terms with respect to $\bar{s}_{l}$ remain since $\mathrm{D}_{s} Z=0$ for any $\mathcal{F}_{t}$-measurable $Z$ and $s \geq t$.
5.2. Proof of Proposition $\mathbf{3 . 1 0}$ - Computations for $\mathfrak{D}_{1}^{n 1}$ and $\mathfrak{D}_{2}^{n 1}$. For $u \in\left\{t, t_{i}\right\}$, the $n$-th Malliavin derivatives of $\psi^{2}\left(V_{u}\right)$ reads

$$
\mathrm{D}_{\boldsymbol{s}_{n}} \psi^{2}\left(V_{u}\right)=\mathrm{D}_{s_{\llbracket 2, n]}} \mathrm{D}_{s_{1}} \psi^{2}\left(V_{u}\right)=\mathrm{D}_{s_{[2, n]}}\left(\left(\psi^{2}\right)^{\prime}\left(V_{u}\right) K\left(u, s_{1}\right)\right)=\cdots=\psi^{2(n)}\left(V_{u}\right) \mathbf{K}\left(u, \boldsymbol{s}_{n}\right),
$$

and therefore $\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \psi^{2}\left(V_{u}\right)\right]=\mathbb{E}\left[\psi^{2(n)}\left(V_{u}\right)\right] \mathbf{K}\left(u, \boldsymbol{s}_{n}\right)$, which, in turn, yields

$$
\begin{aligned}
\mathfrak{D}_{1}^{n 1} & =\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \psi^{2}\left(V_{t}\right)-\mathrm{D}_{\boldsymbol{s}_{n}} \psi^{2}\left(V_{t_{i}}\right)\right]=\mathbb{E}\left[\psi^{2(n)}\left(V_{t}\right)\right] \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)-\mathbb{E}\left[\psi^{2(n)}\left(V_{t_{i}}\right)\right] \mathbf{K}\left(t_{i}, s_{n}\right) \\
& =\mathbb{E}\left[\psi^{2(n)}\left(V_{t}\right)-\psi^{2(n)}\left(V_{t_{i}}\right)\right] \mathbf{K}\left(t, s_{n}\right)+\mathbb{E}\left[\psi^{2(n)}\left(V_{t_{i}}\right)\right]\left(\mathbf{K}\left(t, \boldsymbol{s}_{n}\right)-\mathbf{K}\left(t_{i}, s_{n}\right)\right) .
\end{aligned}
$$

Now, exploiting the regularity assumptions in the statement of Theorem 3.1 together with the fact that $\psi\left(V_{u}\right)$ and its derivatives are bounded in $L^{p}$ for all $p \geq 1$ (thanks to the growth conditions and Lemma 2.21, we can write

$$
\begin{align*}
\left|\mathfrak{D}_{1}^{n 1}\right| & \leq\left|\mathbb{E}\left[\psi^{2(n)}\left(V_{t}\right)-\psi^{2(n)}\left(V_{t_{i}}\right)\right]\right| \mathbf{K}\left(t, s_{n}\right)+\left|\mathbb{E}\left[\psi^{2(n)}\left(V_{t_{i}}\right)\right]\right|\left|\mathbf{K}\left(t, s_{n}\right)-\mathbf{K}\left(t_{i}, s_{n}\right)\right| \\
& \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, s_{n}\right)+\mathbf{C}_{\psi^{2}}\left|\mathbf{K}\left(t, s_{n}\right)-\mathbf{K}\left(t_{i}, s_{n}\right)\right|, \tag{5.6}
\end{align*}
$$

where $\mathbf{C}_{\psi^{2}}$ denotes the $L^{1}$ bound of $\psi^{2(n)}\left(V_{u}\right)$. Let us define the notation

$$
\Delta K\left(t, t_{i}, s\right):=K(t, s)-K\left(t_{i}, s\right), \quad \text { for all } s \in \mathbb{T} .
$$

The first term in (5.6) is given in the proposition, while the second term can be bounded by an iterative procedure in the following sense

$$
\begin{aligned}
\left|\mathbf{K}\left(t, s_{n}\right)-\mathbf{K}\left(t_{i}, s_{n}\right)\right| \leq & K\left(t, s_{n}\right)\left|\mathbf{K}\left(t, s_{n-1}\right)-\mathbf{K}\left(t_{i}, \boldsymbol{s}_{n-1}\right)\right|+\mathbf{K}\left(t_{i}, s_{n-1}\right)\left|\Delta K\left(t, t_{i}, s_{n}\right)\right| \\
\leq & K\left(t, s_{n}\right) K\left(t, s_{n-1}\right)\left|\mathbf{K}\left(t, \boldsymbol{s}_{n-2}\right)-\mathbf{K}\left(t_{i}, s_{n-2}\right)\right| \\
& +K\left(t, s_{n}\right) \mathbf{K}\left(t_{i}, s_{n-2}\right)\left|\Delta K\left(t, t_{i}, s_{n-1}\right)\right|+\mathbf{K}\left(t_{i}, s_{n-1}\right)\left|\Delta K\left(t, t_{i}, s_{n}\right)\right| \\
\leq & \cdots \leq \sum_{l=1}^{n}\left|\Delta K\left(t, t_{i}, s_{l}\right)\right| \mathbf{K}\left(t, s_{\llbracket l+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, l-1 \rrbracket}\right) .
\end{aligned}
$$

Combining the inequalities gives

$$
\left|\mathfrak{D}_{1}^{n 1}\right| \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)+\mathbf{C}_{\psi^{2}} \sum_{l=1}^{n}\left|\Delta K\left(t, t_{i}, s_{l}\right)\right| \mathbf{K}\left(t, s_{\llbracket l+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, \boldsymbol{s}_{\llbracket 1, l-1 \rrbracket}\right)=: \overline{\mathfrak{D}_{1}^{n 1}}\left(t, t_{i}, \boldsymbol{s}_{n}\right)
$$

Similar computations hold for $\psi$ in place of $\psi^{2}$ and yield the desired bound on $\left|\mathfrak{D}_{2}^{n 1}\right|$ :

$$
\left|\mathfrak{D}_{2}^{n 1}\right| \leq \overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)+\mathbf{C}_{\psi} \sum_{l=1}^{n}\left|\Delta K\left(t, t_{i}, s_{l}\right)\right| \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket l+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, \boldsymbol{s}_{\llbracket 1, l-1 \rrbracket}\right)=: \overline{\mathfrak{D}_{2}^{n 1}}\left(t, t_{i}, \boldsymbol{s}_{n}\right)
$$

5.3. Proof of Proposition $\mathbf{3 . 1 0}$ - Computations for $\mathfrak{D}_{1}^{n 2}$. First of all, let us stress that in this section the Malliavin differential operators $\mathcal{D}$ correspond to partitions of the set $\llbracket n \rrbracket$. We write by 5.2 ,

$$
\mathrm{D}_{\boldsymbol{s}_{n}} \partial_{x x} \bar{u}_{t}=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \mid \mathcal{F}_{t}\right]=\mathbb{E}\left[\sum_{k=1}^{n}\left(\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right) \mid \mathcal{F}_{t}\right],
$$

with $\mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)$ defined in (5.4). We have to compute the quantity, defined on Page 18 .

$$
\begin{equation*}
\mathfrak{D}_{1}^{n 2}=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}} \mathbb{E}\left[\phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \mid \mathcal{F}_{t}\right]\right]=\sum_{k=1}^{n} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \underbrace{\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right]}_{\mathfrak{F}_{1}^{k \mathcal{D}}} . \tag{5.7}
\end{equation*}
$$

The following lemma provides a bound for $\left|\mathfrak{F}_{1}^{k \mathcal{D}}\right|$ :
Lemma 5.2. For any fixed $(k, \mathcal{D})$, the bound $\left|\mathfrak{F}_{1}^{k \mathcal{D}}\right| \leq \overline{\mathfrak{F}_{1}^{k \mathcal{D}}}$ holds, with $\overline{\mathfrak{F}_{1}^{k \mathcal{D}}}$ in Appendix $B$,
Combining this with (5.7) yields the bound for $\mathfrak{D}_{1}^{n 2}$ :

$$
\left|\mathfrak{D}_{1}^{n 2}\right| \leq \sum_{k=1}^{n} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \overline{\mathfrak{F}_{1}^{k \mathcal{D}}}\left(t, t_{i}, \boldsymbol{s}_{n}\right)=: \overline{\mathfrak{D}_{1}^{n 2}}\left(t, t_{i}, \boldsymbol{s}_{n}\right)
$$

in Proposition 3.10. The proof of this lemma however requires a decomposition of $\mathfrak{F}_{1}^{k \mathcal{D}}$ followed by a detailed bound analysis.

Proof of Lemma 5.2. The explicit expansion for the $n$-th Malliavin derivative of $X_{T}^{t, \bar{X}_{t}}$ in 5.5 implies

$$
\begin{aligned}
\mathfrak{F}_{1}^{k \mathcal{D}} & =\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right]=\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \prod_{l=1}^{k} \mathrm{D}_{\boldsymbol{I}_{\mathcal{D}}^{l}}^{n_{l}} X_{T}^{t, \bar{X}_{t}}\right] \\
& =\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \prod_{l=1}^{k}\left\{\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{u}\right) \mathbf{K}\left(u, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{u}+\rho \psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}\right] \\
& =\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right) \sum_{\mathcal{U} \subset \llbracket 1, k \rrbracket}\left(\prod_{l \in \mathcal{U}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{u}\right) \mathbf{K}\left(u, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{u}\right)\left(\prod_{l \in \mathcal{U}^{c}} \rho \psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right)\right] \\
& =\sum_{\mathcal{U} \subset \llbracket 1, k \rrbracket} \mathfrak{G}_{1}^{\mathcal{U}} \prod_{l \in \mathcal{U}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right),
\end{aligned}
$$

where the third line simply follows by expanding the product, with

$$
\begin{equation*}
\mathfrak{G}_{1}^{\mathcal{U}}:=\mathbb{E}\left[\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right)\left\{\prod_{l \in \mathcal{U}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{u}\right) \mathbf{K}\left(u, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{u}\right\}\left\{\prod_{l \in \mathcal{U}^{c}} \rho \psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right)\right\}\right] . \tag{5.8}
\end{equation*}
$$

Note that, while it obviously depends on $k, \mathcal{D}$, we omit these dependences from the notations to avoid cluttering. In the beginning, $\mathfrak{F}_{1}^{k \mathcal{D}}$ was also the product of $k+1$ terms; we have just rearranged them here in a different way. Using the bound for $\left|\mathfrak{G}_{1}^{\mathcal{U}}\right|$ provided in Lemma 5.3 below, we obtain
$\left|\mathfrak{F}_{1}^{k \mathcal{D}}\right| \leq \sum_{\mathcal{U} \subset \llbracket 1, k \rrbracket}\left|\mathfrak{G}_{1}^{\mathcal{U}}\right| \prod_{l \in \mathcal{U}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \leq \sum_{\mathcal{U} \subset \llbracket 1, k \rrbracket} \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \frac{T^{H}}{\sqrt{2 H}} \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=2^{k} \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \frac{T^{H}}{\sqrt{2 H}} \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=: \overline{\mathfrak{F}_{1}^{k \mathcal{D}}}$,

Lemma 5.3. For fixed $\mathcal{D}$ (hence fixed $k, \mathcal{U}$ as well), $\left|\mathfrak{G}_{1}^{\mathcal{U}}\right| \leq \overline{\mathfrak{G}_{1}^{\mathcal{U}}}$ holds, for $\mathfrak{G}_{1}^{\mathcal{U}}$ defined in (5.8) and $\overline{\mathfrak{G}_{1}^{\mathcal{U}}}$ in Appendix B.
Proof of Lemma 5.3. Applying Hölder's inequality for $(k+1)$ terms with $|\rho| \leq 1$ yields

$$
\left|\mathfrak{G}_{1}^{\mathcal{U}}\right| \leq \mathbb{E}\left[\left|\phi^{(2+k)}\left(X_{T}^{t, \bar{X}_{t}}\right)\right|^{k+1}\right]^{\frac{1}{k+1}} \prod_{l \in \mathcal{U}} \mathbb{E}\left[\left|\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{u}\right) \mathbf{K}\left(u, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{u}\right|^{k+1}\right]^{\frac{1}{k+1}} \prod_{l \in \mathcal{U}^{c}} \mathbb{E}\left[\left|\psi^{\left(n_{l}-1\right)}\left(V_{\bar{s}_{l}}\right)\right|^{k+1}\right]^{\frac{1}{k+1}}
$$

Since $\phi\left(X_{u}\right), \psi\left(V_{u}\right)$ and all their derivatives are uniformly bounded in $L^{p}$ for any $p \geq 1$ we obtain the following expression that can be further simplified exploiting BDG inequality:

$$
\begin{aligned}
\left|\mathfrak{G}_{1}^{\mathcal{U}}\right| & \leq \mathbf{C}_{\phi, \psi} \prod_{l \in \mathcal{U}} \mathbb{E}\left[\left|\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}\right)}\left(V_{u}\right) \mathbf{K}\left(u, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{u}\right|^{k+1}\right]^{\frac{1}{k+1}} \\
& \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1}^{\frac{\mathcal{U 1}}{k+1}} \prod_{l \in \mathcal{U}}\left\{\left|\int_{\bar{s}_{l}}^{T} \mathbb{E}\left[\psi^{\left(n_{l}\right)}\left(V_{u}\right)^{2}\right] \mathbf{K}\left(u, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} u\right|^{\frac{k+1}{2}}\right\}^{\frac{1}{k+1}} \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \prod_{l \in \mathcal{U}}\left|\int_{\bar{s}_{l}}^{T} \mathbf{K}\left(u, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} u\right|^{\frac{1}{2}},
\end{aligned}
$$

where the constant $\mathbf{C}_{\phi, \psi}$ changes from line to line and only depends on the regularities of $\phi, \psi$ and their derivatives, while $\mathfrak{b}_{k+1}$ represents the BDG constant and we have exploited the fact that $|\mathcal{U}| \leq k+1$. Since $\mathbf{K}\left(u, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \leq$ $K\left(u, s_{n_{l}}\right)^{2} \mathbf{K}\left(s_{n_{l}}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)^{2}$ for $u \in\left[s_{n_{l}}, T\right]$, then

$$
\begin{aligned}
\left|\mathfrak{G}_{1}^{\mathcal{U}}\right| & \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \prod_{l \in \mathcal{U}}\left(\prod_{j \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} K\left(\bar{s}_{l}, s_{j}\right)^{2} \int_{\bar{s}_{l}}^{T} K\left(u, \bar{s}_{l}\right)^{2} \mathrm{~d} u\right)^{\frac{1}{2}} \\
& =\mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \prod_{l \in \mathcal{U}} \prod_{j \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} K\left(s_{n_{l}}, s_{j}\right)\left(\frac{\left(T-\bar{s}_{l}\right)^{2 H}}{2 H}\right)^{\frac{1}{2}} \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{k+1} \frac{T^{H}}{\sqrt{2 H}} \prod_{l \in \mathcal{U}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=: \overline{\mathfrak{G}_{1}^{\mathcal{U}}} .
\end{aligned}
$$

5.4. Proof of Proposition $\mathbf{3 . 1 0}$ - Computations for $\mathfrak{D}_{2}^{n 2}$. First of all, let us emphasise that in this section the Malliavin operators $\mathcal{D}$ correspond to partitions of the set $\mathcal{U}$, while in Section 5.3 they were corresponding to partitions of $\llbracket n \rrbracket$. We start with the following representation for $\mathfrak{D}_{2}^{n 2}$, which helps in the proof:

## Lemma 5.4.

$$
\mathfrak{D}_{2}^{n 2}=\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \int_{t}^{T} \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, r\right)} \mathfrak{F}_{2}^{j \mathcal{U D}}(r) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{U}^{c}}\right) \mathrm{d} r,
$$

where

$$
\mathfrak{F}_{2}^{j \mathcal{U D}}(r):=\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right] .
$$

Proof. First of all, setting

$$
F:=\phi^{\prime \prime}\left(X_{T}^{t, \bar{X}_{t}}\right) \quad \text { and } \quad G:=\int_{t}^{T} \psi^{\prime}\left(V_{r}\right) K(r, t) \mathrm{d} B_{r}
$$

and noticing that for the $n$-th Malliavin derivative of a product,

$$
\mathrm{D}_{\boldsymbol{s}_{n}} \mathbb{E}\left[F G \mid \mathcal{F}_{t}\right]=\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \mathbb{E}\left[\left(\mathrm{D}_{\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}^{|\mathcal{U}|}}^{\mid{ }^{\mid \mathcal{L}}} F\right)\left(\mathrm{D}_{s_{\mathfrak{o}\left(\mathcal{U}^{c}\right)}^{\left|\mathcal{U}^{c}\right|}} G\right) \mid \mathcal{F}_{t}\right]=\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \mathbb{E}\left[\left(\mathrm{D}_{\boldsymbol{s}_{\mathfrak{o}}(\mathcal{U})}^{|\mathcal{U}|} F\right)\left(\mathrm{D}_{s_{\mathfrak{o}}\left(\mathcal{U}^{c}\right)}^{n-|\mathcal{U}|} G\right) \mid \mathcal{F}_{t}\right],
$$

where the ordering map $\mathfrak{o}$ is defined in Section 5.1, and where $0 \leq t_{i} \leq t \leq T$. This implies:

$$
\begin{aligned}
& \mathrm{D}_{\boldsymbol{s}_{n}}\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle \\
& =\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|} \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|}\left(\boldsymbol{s}_{\mathbf{o}(\mathcal{U})}\right)} \mathcal{D} X_{T}^{t, \bar{X}_{t}} \int_{t}^{T} \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{U}^{c}}\right) \mathrm{d} B_{r} \mid \mathcal{F}_{t}\right]
\end{aligned}
$$

$$
=\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|} \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|}\left(s_{\mathbf{o}}(\mathcal{U})\right)} \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}} \int_{t}^{T} \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) K(r, t) \mathbf{K}\left(r, s_{\mathbf{o}(\mathcal{U}}\right) \mathrm{d} B_{r} \mid \mathcal{F}_{t}\right] .
$$

Note that the derivative of $F$ remains a stochastic integral (with no additional term) since $K\left(s_{k}, t\right)=0$ for all $k \leq n$, as $s_{k} \leq t$. By the integration by parts formula,

$$
\begin{aligned}
& \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \int_{t}^{T} \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathfrak{o}\left(\mathcal{U}^{c}\right)}\right) \mathrm{d} B_{r} \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}\right)} \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right] \\
& =\mathbb{E}\left[\int_{t}^{T} \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{U}^{c}}\right)\left(\rho \mathrm{D}_{r}+\bar{\rho} \overline{\mathrm{D}}_{r}\right)\left(\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}\right)} \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right) \mathrm{d} r\right] .
\end{aligned}
$$

Recall from (4.13) that $\overline{\mathrm{D}} X_{T}^{t, \bar{X}_{t}}$ is strictly simpler than $\mathrm{D} X_{T}^{t, \bar{X}_{t}}$ so we focus on the latter and the former can be dealt with in the same way. Note that

$$
\sum_{j=1}^{|\mathcal{U}|}\left(\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|}\left(s_{\mathfrak{o}(\mathcal{U})}\right)} \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right)=\mathrm{D}_{\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}^{|\mathcal{U}|}} \phi^{(2)}\left(X_{T}^{t, \bar{X}_{t}}\right)
$$

and hence

In turn, this can be exploited in order to rewrite the Malliavin derivative above and yields

$$
\begin{aligned}
& \mathfrak{D}_{2}^{n 2}\left(\boldsymbol{s}_{n}\right)=\mathbb{E}\left[\mathrm{D}_{\boldsymbol{s}_{n}}\left\langle\partial_{\omega}\left(\partial_{x} \bar{u}_{t}\right), K^{t}\right\rangle\right] \\
& =\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \int_{t}^{T} \sum_{\mathcal{D} \in \mathcal{D}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathbf{o}(\mathcal{U})}, r\right)} \underbrace{\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right]}_{\mathfrak{F}_{2}^{j \mathcal{U}}(r)} K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{U}^{c}}\right) \mathrm{d} r,
\end{aligned}
$$

as claimed.
The following lemma-proved in Sections 5.4.1-combined with the previous representation for $\mathfrak{D}_{2}^{n 2}$ will then be key to bound the latter:

Lemma 5.5. For any $j, \mathcal{U}, \mathcal{D}$, r, there exist $\left\{\overline{\mathfrak{H}_{1}^{\mathcal{V}}}, \overline{\mathfrak{H}_{2}^{\mathcal{V}}}\right\}_{\mathcal{V}}$, provided in Appendix $B$, such that

$$
\mathfrak{F}_{2}^{j \mathcal{U D}}(r) \leq \sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\left(\prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right) \overline{\mathfrak{H}_{1}^{\mathcal{V}}}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}\right)+\left(\mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right) \overline{\mathfrak{H}_{2}^{\mathcal{V}}}\left(\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}\right)\right\}
$$

where $\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{D}}}:=\left\{\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}: l \in \mathcal{V}\right\}$.
Proof of the bound for $\mathfrak{D}_{2}^{n 2}$ from Proposition 3.10. Putting together Lemma 5.5 and the values of the constants in Appendix B

$$
\begin{aligned}
\left|\tilde{\mathfrak{F}}_{2}^{j \mathcal{U D}}(r)\right| \leq & \sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\left(\prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right) \overline{\mathfrak{H}_{1}^{\mathcal{V}}}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}, s_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}\right),+\left(\mathbf{K}\left(r, s_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right) \overline{\mathfrak{H}_{2}^{\mathcal{V}}}\left(\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}\right)\right\} \\
\leq & \sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{\frac{|\mathcal{V}|+1}{2}} \mathfrak{b}_{j+2} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \cdot \prod_{l \in \mathcal{V}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \cdot \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}}\right)\right. \\
& \left.\quad+\mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{|\mathcal{V}| / 2} \mathfrak{b}_{j+2} \prod_{l \in \mathcal{V}} \mathbf{K}\left(\bar{s}_{l}, s_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(r, s_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}
\end{aligned}
$$

$$
\leq \mathbf{C}_{\phi, \psi}\left(\frac{T^{2 H}}{2 H}\right)^{\frac{j}{2}} \mathfrak{b}_{j+2} 2^{j+1} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \llbracket j-1 \rrbracket} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)
$$

Finally, we exploit this to bound the absolute value of the Malliavin derivative of the 'mixed' derivative of $r$ and obtain

$$
\begin{aligned}
& \left|\mathfrak{D}_{2}^{n 2}\right| \leq \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \int_{t}^{T} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, r\right)}\left|\mathfrak{F}_{2}^{j \mathcal{U} \mathcal{D}}(r)\right| K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathfrak{o}\left(\mathcal{U}^{c}\right)}\right) \mathrm{d} r \\
& \leq \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \int_{t}^{T}\left\{\sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, r\right)} \mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{\frac{j}{2}} \mathfrak{b}_{j+2} 2^{j+1} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathfrak{o}\left(\mathcal{U}^{c}\right)}\right)\right\} \mathrm{d} r .
\end{aligned}
$$

At this point we would like to take the sum over $\mathcal{D}$ out of the integral even though it seems to depend on $r$. In fact, for all $l \in \llbracket j \rrbracket, \mathcal{I}_{\mathcal{D}}^{l}$ does not depend on $r$, and only the structure of $\mathcal{I}_{\mathcal{D}}^{l}$ matters as we can see in the integrand, hence $r$ is a 'mute' variable and we can replace it by any real number $\xi \in \mathbb{T}$. This yields

$$
\begin{aligned}
\left|\mathfrak{D}_{2}^{n 2}\right| & \leq \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{\frac{j}{2}} \mathfrak{b}_{j+2} 2^{j+1} \prod_{l \in \llbracket j-1 \rrbracket} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) K(r, t) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{U}^{c}}\right) \mathrm{d} r \\
& =\sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{\frac{j}{2}} \mathfrak{b}_{j+2} 2^{j+1} \prod_{l \in \llbracket j-1 \rrbracket} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \int_{t}^{T} \mathbf{K}\left(r, s_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r \\
& =\overline{\mathfrak{D}}_{2}^{n 2}\left(t, t_{i}, \boldsymbol{s}_{n}\right) .
\end{aligned}
$$

5.4.1. Proof of Lemma 5.5. We prove the lemma in two steps:

- Step 1 provides a convenient representation for $\mathfrak{F}_{2}^{j \mathcal{U D}}(r)$;
- Step 2 derives the bounds claimed in the lemma.

Step 1: Representation for $\mathfrak{F}_{2}^{j \mathcal{U D}}(r)$. Analogously to what has been done for the analysis of the previous case (see, in particular, Section 5.1, , any derivative $\mathcal{D}$ in $\mathcal{D}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, r\right)$ is associated to a collection of vectors obtained from a non-empty $j$-partition of the set $\left\{s_{i}: i \in \mathcal{U}\right\} \cup\{r\}$. In sight of this and of the computations performed above for the previous case, it is convenient to rewrite the operator $\mathcal{D}$ as

$$
\mathcal{D} X=\mathrm{D}_{\left(\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}, r\right)}^{n_{j}^{\mathcal{U} \mathcal{D}}+1} X \prod_{l=1}^{j-1} \mathrm{D}_{\boldsymbol{I}_{\mathcal{D}}^{l}}^{n_{l}^{\mathcal{U}}} \underset{\mathcal{D}}{ } X
$$

where now we are considering the partition of the set $\mathcal{U}$ that is naturally associated to the partition established above for $\left\{s_{j}: j \in \mathcal{U}\right\} \cup\{r\}$ once $r$ is omitted and we reason on indices, that is

$$
\mathcal{I}_{\mathcal{D}}^{l}=\left(j_{1}^{1}, \cdots, j_{n_{l}^{\mathcal{U}}}^{l}\right), \quad \text { with } n_{l}^{\mathcal{U} \mathcal{D}}:=\left|\mathcal{I}_{\mathcal{D}}^{l}\right|
$$

with $j_{n_{1}}^{1}<\cdots<j_{n_{j-1}}^{j-1}$ and $j_{1}^{l}<\cdots<j_{n_{l}^{\mathcal{D}}}^{l}$, for $l \in \llbracket j \rrbracket$. The motivation for this choice of notation lays in the fact that it enables us to exploit the reasoning for the previous case, as we have $k \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{j}$, but $r$ is not in $\mathcal{I}_{\mathcal{D}}^{j}$ and we have it as an additional kernel $K(w, r)$.

Thus, we write

$$
\begin{aligned}
& \mathfrak{F}_{2}^{j \mathcal{U D}}(r)= \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \mathcal{D} X_{T}^{t, \bar{X}_{t}}\right] \\
&= \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \mathrm{D}_{\left(s_{\mathcal{I}_{\mathcal{D}}^{j}}^{n_{j}^{\mathcal{U}}}+r\right)} X_{T}^{t, \bar{X}_{t}} \prod_{l=1}^{j-1} \mathrm{D}_{\mathcal{I}_{\mathcal{D}}^{l}}^{n_{l}^{\mathcal{U}}} X_{T}^{t, \bar{X}_{t}}\right] \\
&=\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right)\left\{\int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{U}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}+\rho \psi^{\left(n_{j}^{\mathcal{U}}\right.}-1\right)\right. \\
&\left.\left(V_{r}\right) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)\right\} \\
&\left.\cdot \prod_{l=1}^{j-1}\left\{\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{U} \mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}+\rho \psi^{\left(n_{l}^{\mathcal{U} \mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}\right]
\end{aligned}
$$

In the display above since $r \notin \mathcal{I}_{\mathcal{D}}^{j}$ we multiplied by a kernel $K(w, r)$ in the first integrand.

$$
\mathfrak{F}_{2}^{j \mathcal{U D}}(r)=\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right)\left\{\int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{U}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}+\rho \psi^{\left(n_{j}^{\mathcal{U}}\right.}-1\right)\left(V_{r}\right) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)\right\}
$$

$$
\begin{aligned}
& \left.\cdot \sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left(\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right)\left(\prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{\mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}}\right)\right)\right] \\
& =\sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\mathbb { E } \left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{D}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right.\right. \\
& \left.\cdot\left(\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mu \mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right)\left(\prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{\mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}}\right)\right)\right] \\
& +\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \rho \psi^{\left(n_{j}^{\mathcal{D}}-1\right)}\left(V_{r}\right) \mathbf{K}\left(r, s_{\mathcal{I}_{\mathcal{D}}^{j}}\right)\right. \\
& \left.\left.\cdot\left(\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right)\left(\prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{\mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}}\right)\right)\right]\right\} \\
& =\sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\mathbb { E } \left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{D}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right.\right. \\
& \left.\cdot\left(\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}\right)\left(\prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{\mu \mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right)\right)\right] \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \\
& +\mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \rho \psi^{\left(n_{j}^{\mathcal{U}}\right.}-1\right)\left(V_{r}\right)\left(\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right) \\
& \left.\left.\cdot \prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{U \mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right)\right] \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \\
& =: \sum_{\mathcal{V} \subset \llbracket j-1 \rrbracket}\left\{\mathfrak{H}_{1}^{\mathcal{V}}(r)\left(\prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right)+\mathfrak{H}_{2}^{\mathcal{V}}(r)\left(\mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}^{c}} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right)\right\} .
\end{aligned}
$$

Step 2: Estimates for $\mathfrak{H}_{1}^{\mathcal{V}}(r)$ and $\mathfrak{H}_{2}^{\mathcal{V}}(r)$
Applying Hölder inequality on $j+2$ terms yield and $|\rho| \leq 1$,

$$
\begin{aligned}
\mathfrak{H}_{1}^{\mathcal{V}}(r)=\mathbb{E} & {\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-\mid \mathcal{U l |})}\left(V_{r}\right) \int_{r}^{T} \psi^{\left(n_{j}^{u \mathcal{D}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, s_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right.} \\
& \left.\cdot \prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{u \mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w} \prod_{l \in \mathcal{V} c} \rho \psi^{\left(n_{l}^{u \mathcal{D}}-1\right)}\left(V_{\bar{S}_{l}}\right)\right] \\
\leq \mathbb{E} & {\left[\left|\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right)\right|^{j+2}\right]^{\frac{1}{j+2}} \mathbb{E}\left[\left|\psi^{(1+n-\mid \mathcal{U | |})}\left(V_{r}\right)\right|^{j+2}\right]^{\frac{1}{j+2}} } \\
& \cdot \mathbb{E}\left[\left|\int_{r}^{T} \psi^{\left(n_{j}^{u \mathcal{D}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, s_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right|^{j+2}\right]^{\frac{1}{j+2}} \\
& \cdot \prod_{l \in \mathcal{V}} \mathbb{E}\left[\left|\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{u \mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, s_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}\right|^{j+2}\right]^{\frac{1}{j+2}} \prod_{l \in \mathcal{V}^{c}} \mathbb{E}\left[\left|\psi^{\left(n_{l}^{\mathcal{D}}-1\right)}\left(V_{\bar{s}_{l}}\right)\right|^{j+2}\right]^{\frac{1}{j+2}}
\end{aligned}
$$

Since $\phi\left(X_{r}\right), \psi\left(V_{r}\right)$ and all their derivatives are uniformly bounded in $L^{p}$ for any $p \geq 1$, we obtain the following expression that can be further simplified exploiting BDG inequality:

$$
\begin{aligned}
\left|\mathfrak{H}_{1}^{\mathcal{V}}(r)\right| \leq & \mathbf{C}_{\phi, \psi} \mathbb{E}\left[\left|\int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{U}}\right)}\left(V_{w}\right) K(w, r) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \mathrm{d} B_{w}\right|^{j+2}\right]^{\frac{1}{j+2}} \\
& \cdot \prod_{l \in \mathcal{V}} \mathbb{E}\left[\left|\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{U}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}\right|^{j+2}\right]^{\frac{1}{j+2}} \\
& \leq \mathbf{C}_{\phi, \psi} \frac{\mathfrak{b}_{j+2}}{\frac{|\mathcal{V}|+1}{j+2}} \mathbb{E}\left[\left(\int_{r}^{T} \psi^{\left(n_{j}^{\mathcal{D}}\right)}\left(V_{w}\right)^{2} K(w, r)^{2} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)^{2} \mathrm{~d} w\right)^{\frac{j+2}{2}}\right]^{\frac{1}{j+2}}
\end{aligned}
$$

$$
\begin{aligned}
& \quad \cdot \mathfrak{b}_{j+2} \prod_{l \in \mathcal{V}} \mathbb{E}\left[\left(\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{U \mathcal{D}}\right)}\left(V_{w}\right)^{2} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} w\right)^{\frac{j+2}{2}}\right]^{\frac{1}{j+2}} \\
& \leq \mathbf{C}_{\phi, \psi} \frac{\mathfrak{b}_{j+2} \frac{|\mathcal{V}|+1}{j+2}}{} \mathbb{E}\left[\sup _{w \in \mathbb{T}}\left|\psi^{\left(n_{j}^{U \mathcal{D}}\right)}\left(V_{w}\right)\right|^{j+2}\right]^{\frac{1}{j+2}}\left(\int_{r}^{T} K(w, r)^{2} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}} \\
& \quad \prod_{l \in \mathcal{V}} \mathbb{E}\left[\sup _{w \in \mathbb{T}}\left|\psi^{\left(n_{j}^{U \mathcal{D}}\right)}\left(V_{w}\right)\right|^{j+2}\right]^{\frac{1}{j+2}}\left(\int_{\bar{s}_{l}}^{T} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}} \\
& \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{j+2}\left(\int_{r}^{T} K(w, r)^{2} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}} \prod_{l \in \mathcal{V}}\left(\int_{\bar{s}_{l}}^{T} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}}
\end{aligned}
$$

where the constant $\mathbf{C}_{\phi, \psi}$ changes from line to line and only depends on the regularity of $\phi, \psi$ and their derivatives, while $\mathfrak{b}_{j+2}$ is the BDG constant; in the last line we used $|\mathcal{V}| \leq j-1$. Finally, since

- for $w \in[r, T], 0 \leq K\left(w, s_{k}\right) \leq K\left(r, s_{k}\right)$, for all $k \in \mathcal{I}_{\mathcal{D}}^{j}$;
- for $w \in\left[\bar{s}_{l}, T\right], \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \leq K\left(w, \bar{s}_{l}\right)^{2} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)^{2}$;
we obtain

$$
\begin{aligned}
\left|\mathfrak{H}_{1}^{\mathcal{V}}(r)\right| & \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{j+2} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right)\left|\int_{r}^{T} K(w, r)^{2} \mathrm{~d} w\right|^{\frac{1}{2}} \prod_{l \in \mathcal{V}}\left(\mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left|\int_{\bar{s}_{l}}^{T} K\left(w, \bar{s}_{l}\right)^{2} \mathrm{~d} w\right|^{\frac{1}{2}}\right) \\
& \leq \mathbf{C}_{\phi, \psi}\left(\frac{T^{2 H}}{2 H}\right)^{\frac{|\mathcal{V}|+1}{2}} \mathfrak{b}_{j+2} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}\right) \prod_{l \in \mathcal{V}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=: \overline{\mathfrak{H}}_{1}^{\mathcal{V}}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j}}, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}}\right)
\end{aligned}
$$

where $\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}:=\left\{\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}, l \in \mathcal{V}\right\}$.
Reasoning analogously for $\mathfrak{H}_{2}^{\mathcal{V}}(r)$, namely through application of the Hölder inequality on $j+2$ terms and an application of the BDG inequality, we obtain

$$
\begin{aligned}
& \mathfrak{H}_{2}^{\mathcal{V}}(r)= \mathbb{E}\left[\phi^{(2+j)}\left(X_{T}^{t, \bar{X}_{t}}\right) \psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right) \psi^{\left(n_{j}^{\mathcal{U}}\right.}-1\right) \\
&\left.V_{r}\right)\left.\prod_{l \in \mathcal{V}} \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{U}} \mathcal{D}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w} \prod_{l \in \mathcal{V}^{c}} \rho \psi^{\left(n_{l}^{\mathcal{U D}}\right.}-1\right) \\
& \leq\left.\mathbb{E}\left[\mid \phi_{\bar{s}_{l}}\right)\right] \\
&\left.\left.\cdot \prod_{l \in \mathcal{V}} \mathbb{E}\left[\left|\int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{U} \mathcal{D}}\right)}\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}\right|_{T}^{t, \bar{X}_{t}}\right)\right|^{j+2}\right]^{\frac{1}{j+2}} \mathbb{E}\left[\left|\psi^{(1+n-|\mathcal{U}|)}\left(V_{r}\right)\right|^{j+2}\right]_{l \in \mathcal{V}^{c}}^{\frac{1}{j+2}} \mathbb{E}\left[\left|\psi^{\left(n_{j}^{\mathcal{U}}-1\right)}\left(V_{r}\right)\right|^{j+2}\right]^{\frac{1}{j+2}} \\
& \mathbb{E}\left[\left|\psi^{\left(n_{l}^{\mathcal{U}}-1\right)}\left(V_{\bar{s}_{l}}\right)\right|^{j+2}\right]^{\frac{1}{j+2}} \cdot
\end{aligned}
$$

Now, since $\phi\left(X_{r}\right), \psi\left(V_{r}\right)$ and all their derivatives are uniformly bounded in $L^{p}$ for any $p \geq 1$, we obtain the following expression that can be further simplified exploiting BDG inequality:

$$
\begin{aligned}
\left|\mathfrak{H}_{2}^{\mathcal{V}}(r)\right| & \left.\leq\left.\mathbf{C}_{\phi, \psi} \prod_{l \in \mathcal{V}} \mathbb{E}\left[\mid \int_{\bar{s}_{l}}^{T} \psi^{\left(n_{l}^{\mathcal{U}}\right.}\right)\left(V_{w}\right) \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \mathrm{d} B_{w}\right|^{j+2}\right]^{\frac{1}{j+2}} \\
& \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{j+2}^{\frac{|\mathcal{V}|}{j+2}} \prod_{l \in \mathcal{V}} \mathbb{E}\left[\sup _{w \in \mathbb{T}}\left|\psi^{\left(n_{l}^{\mathcal{U}}\right)}\left(V_{w}\right)\right|^{j+2}\right]^{\frac{1}{j+2}}\left(\int_{\bar{s}_{l}}^{T} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}} \\
& \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{j+2} \prod_{l \in \mathcal{V}}\left(\int_{\bar{s}_{l}}^{T} \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \mathrm{~d} w\right)^{\frac{1}{2}}
\end{aligned}
$$

Exploiting again that, for $w \in\left[s_{n_{l}}, T\right], \mathbf{K}\left(w, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)^{2} \leq K\left(w, \bar{s}_{l}\right)^{2} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)^{2}$, we obtain

$$
\left|\mathfrak{H}_{2}^{\mathcal{V}}(r)\right| \leq \mathbf{C}_{\phi, \psi} \mathfrak{b}_{j+2} \prod_{l \in \mathcal{V}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left|\int_{\bar{s}_{l}}^{T} K\left(w, \bar{s}_{l}\right)^{2} \mathrm{~d} w\right|^{\frac{1}{2}} \leq \mathbf{C}_{\phi, \psi}\left|\frac{T^{2 H}}{2 H}\right|^{\frac{|\mathcal{V}|}{2}} \mathfrak{b}_{j+2} \prod_{l \in \mathcal{V}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=: \overline{\mathfrak{H}}_{2}^{\mathcal{V}}\left(\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{\mathcal{V}}}\right)
$$

6. Proof of Proposition $3.10-$ THE $\left|\mathfrak{C}_{1}^{n}(t)\right|$ AND $\left|\mathfrak{C}_{2}^{n}(t)\right|$ TERMS
6.1. Computations for $\left|\mathfrak{C}_{1}^{n}(t)\right|$. The following representation for a bound on $\left|\mathfrak{C}_{1}^{n}(t)\right|$ helps us split the steps of the proof:

Lemma 6.1. The following bound holds for any $t \in\left[t_{i}, t_{i+1}\right)$ :

$$
\left|\mathfrak{C}_{1}^{n}(t)\right| \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathfrak{M}_{1}^{k \mathcal{D}}+\sum_{m=1}^{n} \mathfrak{N}_{1}^{k \mathcal{D}}(m)\right\}
$$

where

$$
\begin{aligned}
\mathfrak{M}_{1}^{k \mathcal{D}} & :=\int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathrm{d} \boldsymbol{s}_{n}, \\
\mathfrak{N}_{1}^{k \mathcal{D}}(m) & :=\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, \boldsymbol{s}_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathrm{d} \boldsymbol{s}_{n} .
\end{aligned}
$$

Proof of Lemma 6.1. Introduce the following decomposition and terms:

$$
\begin{aligned}
\overline{\mathfrak{D}_{1}^{n 1}} & =\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)+\mathbf{C}_{\psi^{2}} \mathfrak{J}\left(t, t_{i}, \boldsymbol{s}\right), \\
\mathfrak{J}\left(t, t_{i}, \boldsymbol{s}\right) & :=\sum_{l=1}^{n}\left|\Delta K\left(t, t_{i}, s_{l}\right)\right| \mathbf{K}\left(t, s_{\llbracket l+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, l-1 \rrbracket}\right), \\
\mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) & :=\sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}(s)} \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) .
\end{aligned}
$$

The functions $\mathfrak{J}\left(t, t_{i}, \boldsymbol{s}_{n}\right)$ and $\mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right)$ are symmetric in their last argument and hence their product is symmetric in $\boldsymbol{s}_{n}$ as well. With the definitions of $\mathfrak{D}_{1}^{n 1}$ and $\mathfrak{D}_{1}^{n 2}$ on Page 18 we can rewrite $\mathfrak{C}_{1}^{n}(t)$ in (3.7) as

$$
\begin{aligned}
\left|\mathfrak{C}_{1}^{n}(t)\right| & =\left|\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \mathfrak{D}_{1}^{n 1}\left(\boldsymbol{s}_{n}\right) \mathfrak{D}_{1}^{n 2}\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}\right| \leq \int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}}\left|\mathfrak{D}_{1}^{n 1}\left(\boldsymbol{s}_{n}\right)\right|\left|\mathfrak{D}_{1}^{n 2}\left(\boldsymbol{s}_{n}\right)\right| \mathrm{d} \boldsymbol{s}_{n} \\
& \leq \mathbf{C}_{\phi, \psi} \frac{T^{H}}{\sqrt{2 H}} \int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \overline{\mathfrak{D}_{1}^{n 1}} \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n} \\
& =\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \int_{0}^{t} \int_{0}^{t} \cdots \int_{0}^{t} \overline{\mathfrak{D}_{1}^{n 1}} \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n},
\end{aligned}
$$

where the last equality stems from the symmetry of $\mathfrak{K}\left(t, t_{i}, \cdot\right)$ and Lemma A. 1 This is crucial in the following in order to get the convergence of the terms in the series. Let us start with a complete discussion of $\left|\mathfrak{C}_{1}^{n}(t)\right|$. We can now write

$$
\begin{align*}
\left|\mathfrak{C}_{1}^{n}(t)\right| & \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \int_{0}^{t} \int_{0}^{t} \cdots \int_{0}^{t}\left[\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)+\mathbf{C}_{\psi^{2}} \mathfrak{J}\left(t, t_{i}, \boldsymbol{s}_{n}\right)\right] \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n} \\
& =\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}+\sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \int_{[0, t]^{n}} \mathfrak{J}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathfrak{K}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}\right\} \\
& =\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}}\{\overline{\mathbf{C}\left(t^{\gamma}-t_{i}^{\gamma}\right) \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \underbrace{\int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}}\right) \mathrm{d} \boldsymbol{s}_{n}}_{\mathfrak{M}_{10}^{k \mathcal{D}}}}  \tag{6.1}\\
& +\sum_{m=1}^{n} \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \underbrace{\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, \boldsymbol{s}_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\boldsymbol{\mathcal { D }}}}\right) \mathrm{d} \boldsymbol{s}_{n}}_{\mathfrak{N}_{1}^{k \mathcal{D}}(m)}\} \tag{6.2}
\end{align*}
$$

Notice that here, in order to keep track of the Malliavin differential operator considered, we write $\mathcal{I}_{\mathcal{D}}^{l}$. Finally, taking the summations out of integrals we obtain the desired expression.

We now provide bounds for both $\mathfrak{M}_{1}^{k \mathcal{D}}$ and $\mathfrak{N}_{1}^{k \mathcal{D}}$-proved in Sections 6.1.1 and 6.1 .2 below-before proving the estimate for $\left|\mathfrak{C}_{1}^{n}(t)\right|$. We introduce for convenience the quantities

$$
\beta_{l}:=\mathrm{B}\left(2 H\left(n_{l}-1\right)+1, H_{+}\right) \quad \text { and } \quad \beta_{+}:=\mathrm{B}\left(H_{+}, H_{+}\right)
$$

where B denotes the Beta function. Since the first argument of $\mathrm{B}(\cdot)$ is always greater than $1-2 H>0$, then there exists $\bar{\beta}>0$ such that $\beta_{l} \leq \bar{\beta}$ for any $l$. We shall further write $\beta_{l}^{\mathcal{U D}}$ (and similar) whenever $n_{l}$ is replaced by $n_{l}^{\mathcal{U} \mathcal{D}}$.

Lemma 6.2. For any fixed $(n, k, \mathcal{D})$, we have

$$
\mathfrak{M}_{1}^{k \mathcal{D}} \leq \overline{\mathfrak{M}_{1}^{k \mathcal{D}}} \quad \text { and } \quad \mathfrak{N}_{1}^{k \mathcal{D}}(m) \leq \begin{cases}{ }_{1} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{H_{+}}, & \text {if } m \in{ }_{1} \mathcal{N}_{1} \\ { }_{2} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{3 H+\frac{1}{2}}, & \text { if } m \in{ }_{2} \mathcal{N}_{1}\end{cases}
$$

where the constants are provided in Appendix $B$ and

$$
{ }_{1} \mathcal{N}_{1}:=\bigcup_{l=1}^{k}\left\{j_{n_{l}}^{l}\right\}, \quad \text { and } \quad{ }_{2} \mathcal{N}_{1}:=\bigcup_{l=1}^{k} \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}
$$

and such that ${ }_{1} \mathcal{N}_{1} \cup_{2} \mathcal{N}_{1}=\llbracket n \rrbracket$.
Exploiting these bounds for $\mathfrak{M}_{1}^{k \mathcal{D}}$ and $\mathfrak{N}_{1}^{k \mathcal{D}}(m)$, we can then prove the estimate for $\left|\mathfrak{C}_{1}^{n}(t)\right|$ in Proposition 3.10 .

Proof of Proposition 3.10 - Bound on $\left|\mathfrak{C}_{1}^{n}(t)\right|$. From Lemmas 6.1 and 6.2 , we can write (details of identical indices from line to line are not repeated)

$$
\begin{aligned}
\left|\mathfrak{C}_{1}^{n}(t)\right| & \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \mathfrak{M}_{1}^{k \mathcal{D}}+\sum_{m=1}^{n} \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \sum_{\mathcal{D} \in \mathcal{D}_{k}^{n}\left(\boldsymbol{s}_{n}\right)} \mathfrak{N}_{1}^{k \mathcal{D}}(m)\right\} \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \mathfrak{b}_{k+1} 2^{k}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \overline{\mathfrak{M}_{1}^{k \mathcal{D}}}+\sum_{m \in_{1} \mathcal{N}_{1}} 1 \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{H+\frac{1}{2}}+\sum_{m \in_{2} \mathcal{N}_{1}}{ }_{2} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{3 H+\frac{1}{2}}\right\}
\end{aligned}
$$

Now, by definition of these constants we have

$$
\begin{align*}
\overline{\mathbf{C}} \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \mathfrak{b}_{k+1} 2^{k} \overline{\mathfrak{M}_{1}^{k \mathcal{D}}} & \leq \overline{\mathbf{C}} \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \mathfrak{b}_{k+1} 2^{k}\left|\frac{t^{2 H(n-k)+k H_{+}}}{(2 H)^{n-k}} \prod_{l=1}^{k} \beta_{l}\right| \\
& \leq \overline{\mathbf{C}} \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H}\left|\max _{k=1, \cdots, n} \bar{\beta}^{k}\right| \sum_{k} \mathfrak{b}_{k+1} 2^{k} S(n, k)=: \overline{\overline{\mathfrak{M}_{1}^{n}}} \tag{6.3}
\end{align*}
$$

moreover,

$$
\begin{align*}
& \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \sum_{m \in_{1} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}{ }_{1} \overline{\mathfrak{N}}_{1}^{k \mathcal{D}}(m) \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \sum_{m \in_{1} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}\left|\mathfrak{g}_{0} \frac{t^{2 H(n-k+1)+(k-1) H_{+}}}{(2 H)^{n-k+1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l}\right| \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{0} \sum_{k, \mathcal{D}} \sum_{m \in_{1} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}\left|\prod_{l=1, \neq l^{m}}^{k} \beta_{l}\right| \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{0} \sum_{k, \mathcal{D}} \sum_{m \in_{1} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k} \bar{\beta}^{k} \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{0} \sum_{k} \sum_{m \in_{1} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k} S(n, k) \bar{\beta}^{k} \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{0} n \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} S(n, k) \bar{\beta}^{k} \\
& \left.\leq\left.\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{0} n\right|_{k=1, \ldots, n} ^{\max ^{2 H}} \bar{\beta}^{k} \right\rvert\, \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} S(n, k)=: 1_{1} \overline{\overline{\mathfrak{N}_{1}^{n}}} \tag{6.4}
\end{align*}
$$

and similarly

$$
\begin{align*}
& \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}{ }_{2} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \\
& \left.\leq\left.\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \sum_{k, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}\right|_{\mathfrak{g}_{2 H}} \beta_{+} \frac{t^{2 H(n-k-1)+(k-1) H_{+}}}{(2 H)^{n-k-1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l} \right\rvert\, \\
& \left.\leq\left.\frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{2 H} \beta_{+} \sum_{k, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k}\right|_{l=1, \neq l^{m}} ^{k} \beta_{l} \right\rvert\, \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{2 H} \beta_{+} \sum_{k, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{1}} \mathfrak{b}_{k+1} 2^{k} \bar{\beta}^{k} \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{2 H} \beta_{+} n \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} \bar{\beta}^{k} S(n, k) \\
& \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H}}{\sqrt{2 H}} \frac{T^{\left(2 H+\frac{1}{2}\right) n}}{2 H} \mathfrak{g}_{2 H} \beta_{+} n\left|\max _{k=1, \ldots, n} \bar{\beta}^{k}\right| \sum_{k=1}^{n} \mathfrak{b}_{k+1} 2^{k} S(n, k)=: 2_{2} \overline{\overline{\mathfrak{N}_{1}^{n}}} \tag{6.5}
\end{align*}
$$

6.1.1. Proof of Lemma 6 6.2 - Bound on $\mathfrak{M}_{1}^{k \mathcal{D}}$. We want to integrate the kernels in a particular order hence we separate them as $\mathbf{K}\left(t, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right)=K\left(t, \bar{s}_{l}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)$. The bound on $\mathfrak{M}_{1}^{k \mathcal{D}}$, defined in 6.1), then follows from the computations:

$$
\mathfrak{M}_{1}^{k \mathcal{D}}=\int_{[0, t]^{n}} \prod_{l=1}^{k} K\left(t, \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathrm{d} \boldsymbol{s}_{n}=\prod_{l=1}^{k} \int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left(\prod_{j \in \tilde{\mathcal{I}}_{\mathcal{D}}^{l}} \int_{0}^{t} K\left(\bar{s}_{l}, s_{j}\right) K\left(t, s_{j}\right) \mathrm{d} s_{j}\right) \mathrm{d} \bar{s}_{l}
$$

The linearity of the integrals and since $K\left(t, s_{j}\right)$ depends on different variables allowed us to interchange integral and product. We use the same technique repeatedly in the remainder of the article. Exploiting $K\left(t, s_{j}\right) \leq K\left(\bar{s}_{l}, s_{j}\right)$ yields

$$
\begin{aligned}
\mathfrak{M}_{1}^{k \mathcal{D}} & \leq \prod_{l=1}^{k} \int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left(\prod_{j \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} \int_{0}^{\bar{s}_{l}} K\left(\bar{s}_{l}, s_{j}\right)^{2} \mathrm{~d} s_{j}\right) \mathrm{d} \bar{s}_{l} \\
& =\prod_{l=1}^{k} \int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left|\frac{\bar{s}_{l}^{2 H}}{2 H}\right|^{n_{l}-1} \mathrm{~d} \bar{s}_{l}=(2 H)^{k-n} \prod_{l=1}^{k} \int_{0}^{t} K\left(t, \bar{s}_{l}\right) \bar{s}_{l}^{2 H\left(n_{l}-1\right)} \mathrm{d} \bar{s}_{l}=\frac{t^{2 H(n-k)+k H_{+}}}{(2 H)^{n-k}} \prod_{l=1}^{k} \beta_{l}=: \overline{\mathfrak{M}_{1}^{k \mathcal{D}}}
\end{aligned}
$$

since the sets $\left(\mathcal{I}_{\mathcal{D}}^{l}\right)_{l=1}^{k}$ form a partition of $\llbracket n \rrbracket$ and thus $\sum_{l=1}^{k} n_{l}=n$.
6.1.2. Proof of Lemma 6.2 - Bound on $\mathfrak{N}_{1}^{k \mathcal{D}}$. Fix $m \in \llbracket n \rrbracket$ and denote by $\widetilde{\mathcal{I}}_{\mathcal{D}}^{m}$ the set of indices it belongs to. We have to study separately two cases:
(1) $m \in{ }_{2} \mathcal{N}_{1}$, that is $m \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}$, with $l \in \llbracket 1, k \rrbracket$;
(2) $m \in{ }_{1} \mathcal{N}_{1}$ that is $m=j_{n_{l}}^{l}$ for some $l \in \llbracket 1, k \rrbracket$ or $m=j_{n_{l} m}^{l^{m}}$.

To perform the proof, we need to integrate the kernels in a particular order, however both cases consist of different kernels and therefore yield different results. Recall the definition from 6.2

$$
\begin{equation*}
\mathfrak{N}_{1}^{k \mathcal{D}}(m):=\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathrm{d} \boldsymbol{s}_{n} \tag{6.6}
\end{equation*}
$$

We rewrite the integrand in $\mathfrak{N}_{1}^{k \mathcal{D}}(m)$ reordering and grouping terms. More precisely, we split the kernels into two groups depending on the second argument: (i) $s_{i}$ with $i \in \mathcal{I}_{\mathcal{D}}^{l}$ with $l \neq l^{m}$, (ii) $s_{i}$ with $i \in \mathcal{I}_{\mathcal{D}}^{l}$ with $l=l^{m}$. Furthermore, we separate the terms where $s_{i} \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}$ and $s_{i}=\bar{s}_{l}$, corresponding to ${ }_{2} \mathcal{N}_{1}$ and ${ }_{1} \mathcal{N}_{1}$ respectively. This is illustrated in the following expression, where we denote $\mathfrak{m}\left(s_{j}\right)=t_{i}$, if $j<m$, and $\mathfrak{m}\left(s_{j}\right)=t$, if $j>m$.

$$
\begin{equation*}
\mathbf{K}\left(t, \boldsymbol{s}_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket 1, m-1 \rrbracket}\right)=\prod_{l=1, i \neq l^{m}}^{k} \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{l}}\right) \cdot \prod_{i \in \mathcal{I}_{\mathcal{D}}^{l m} \backslash\{m\}} K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right) \tag{6.7}
\end{equation*}
$$

$$
=\prod_{l=1, i \neq l^{m}}^{k}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \cdot K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right) \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l m} \backslash\{m\}} K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right) .
$$

On the other hand we have

$$
\begin{equation*}
\prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)=\left\{\prod_{l=1, \neq l^{m}}^{k} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}\left\{\prod_{i \in \mathcal{I}_{\mathcal{D}}^{l m} \backslash\{m\}} K\left(\bar{s}_{l^{m}}, s_{i}\right)\right\} \tag{6.8}
\end{equation*}
$$

In Case (1), the term $K\left(\bar{s}_{l^{m}}, s_{m}\right)$ exists in the second product of 6.8 and can be isolated, while it cannot be done in Case (2); this is the main difference that yields to different error rates.

Case (1). Applying the representations 6.7 and 6.8 leads to the representation

$$
\begin{aligned}
& \left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{k} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \\
& =\prod_{l=1, i \neq l^{m}}^{k}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \\
& \quad K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left\{\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l^{m}} \backslash\{m\}}\left[K\left(\bar{s}_{l^{m}}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right]\right\},
\end{aligned}
$$

Note that when integrating against $s_{i}$, they become mute variables but we keep the indices for clarity. Thus we obtain

$$
\begin{aligned}
& \mathfrak{N}_{1}^{k \mathcal{D}}(m) \leq \prod_{l=1, \neq l^{m}}^{k} \int_{0}^{t}\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \underbrace{\left[\int_{0}^{\bar{s}_{l}} \cdots \int_{0}^{\bar{s}_{l}} \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\left[K\left(\bar{s}_{l}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right] \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right]}_{\mathfrak{R}_{l}\left(\bar{s}_{l}\right)}\} \mathrm{d} \bar{s}_{l} . \\
& \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left\{\int_{0}^{\bar{s}_{l^{m}}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} s_{m} .\right.
\end{aligned}
$$

$$
\begin{aligned}
& =\prod_{l=1, \neq l^{m}}^{k} \underbrace{\int_{0}^{t}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \Re_{l}\left(\bar{s}_{l}\right)\right\} \mathrm{d} \bar{s}_{l}}_{\mathfrak{P}_{l}(t)} \cdot \underbrace{\int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left\{\int_{0}^{\bar{s}_{l^{m}}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} s_{m}\right\} \mathfrak{S}_{l^{m}}\left(\bar{s}_{l^{m}}\right) \mathrm{d} \bar{s}_{l^{m}}}_{\mathfrak{Q}(t)},
\end{aligned}
$$

where the following simplifications for $\Re_{l}\left(\bar{s}_{l}\right)$ and $\mathfrak{S}_{l^{m}}\left(\bar{s}_{l^{m}}\right)$ are straightforward (recall that, here, the index $i$ is fixed and omitted from the notations):

$$
\Re_{l}(s)=\mathfrak{R}(s)^{n_{l}-1} \quad \text { and } \quad \mathfrak{S}_{l}(s)=\mathfrak{R}(s)^{n_{l}-2}
$$

where

$$
\mathfrak{R}(s):=\int_{0}^{s} K\left(s, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right) \mathrm{d} s_{i}
$$

We now compute separately the contributions of the different terms that make up the integrand; for this purpose we exploit repeatedly the results gathered in Lemma A. 2 An application of A.3 yields

$$
\begin{equation*}
\mathfrak{S}_{l}(s) \leq\left(\frac{s^{2 H}}{2 H}\right)^{n_{l}-2} \quad \text { and } \quad \Re_{l}(s) \leq\left(\frac{s^{2 H}}{2 H}\right)^{n_{l}-1} \tag{6.9}
\end{equation*}
$$

The last inequality in turn yields

$$
\begin{equation*}
\mathfrak{P}_{l}(t) \leq \frac{1}{(2 H)^{n_{l}-1}} \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \bar{s}_{l}^{2 H\left(n_{l}-1\right)} \mathrm{d} \bar{s}_{l} \leq \frac{\beta_{l}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}} \tag{6.10}
\end{equation*}
$$

Indeed, if $\mathfrak{m}\left(\bar{s}_{l}\right)=t$, then

$$
\mathfrak{P}_{l}(t) \leq \frac{1}{(2 H)^{n_{l}-1}} \int_{0}^{t} K\left(t, \bar{s}_{l}\right) \bar{s}_{l}^{2 H\left(n_{l}-1\right)} \mathrm{d} \bar{s}_{l}=\frac{\beta_{l}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}}
$$

Otherwise, if $\mathfrak{m}\left(\bar{s}_{l}\right)=t_{i} \leq t$, we have

$$
\begin{aligned}
\mathfrak{P}_{l}(t) & \leq \frac{1}{(2 H)^{n_{l}-1}} \int_{0}^{t} K\left(t_{i}, \bar{s}_{l}\right) \bar{s}_{l}^{2 H\left(n_{l}-1\right)} \mathrm{d} \bar{s}_{l}=\frac{1}{(2 H)^{n_{l}-1}} \int_{0}^{t_{i}} K\left(t_{i}, \bar{s}_{l}\right) \bar{s}_{l}^{2 H\left(n_{l}-1\right)} \mathrm{d} \bar{s}_{l} \\
& =\frac{\beta_{l}}{(2 H)^{n_{l}-1}} t_{i}^{2 H\left(n_{l}-1\right)+H_{+}} \leq \frac{\beta_{l}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}}
\end{aligned}
$$

Now, exploiting 6.9, we can write

$$
\begin{align*}
\mathfrak{Q}(t) & =\int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left[\int_{0}^{\bar{s}_{l^{m}}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} s_{m} \widetilde{S}_{l^{m}}\left(\bar{s}_{l^{m}}\right)\right] \mathrm{d} \bar{s}_{l^{m}} \\
& \leq \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left[\int_{0}^{\bar{s}_{l^{m}}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} s_{m}\left(\frac{\bar{s}_{l^{m}}^{2 H}}{2 H}\right)^{n_{l^{m}}-2}\right] \mathrm{d} \bar{s}_{l^{m}} \\
& =\frac{1}{(2 H)^{n_{l} m-2}} \int_{0}^{t}\left[\int_{0}^{\bar{s}_{l^{m}}} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \bar{s}_{l^{m}}^{2 H\left(n_{l} m-2\right)} \mathrm{d} s_{m}\right] \mathrm{d} \bar{s}_{l^{m}} \\
& \leq \frac{t^{2 H\left(n_{l}^{m}-2\right)}}{(2 H)^{n_{l}^{m-2}}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left[\int_{s_{m}}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right) K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} \bar{s}_{l^{m}}\right] \mathrm{d} s_{m} \\
& \leq \beta_{+} \frac{t^{2 H\left(n_{l}^{m}-2\right)}}{(2 H)^{n_{l} m-2}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left(t-s_{m}\right)^{2 H} \mathrm{~d} s_{m} \tag{6.11}
\end{align*}
$$

where the last inequality follows from A.2). Recall that Case (1) implies $\bar{s}_{l} \neq s_{m}$ : this allows to isolate the kernel $K\left(\bar{s}_{l}, s_{m}\right)$ which in turn engenders a smoothing function in the last integral 6.11. Exploiting the bounds in 6.10) and 6.11), we obtain

$$
\begin{aligned}
\mathfrak{N}_{1}^{k \mathcal{D}}(m) \leq \mathfrak{Q}(t) \prod_{l=1, \neq l^{m}}^{k} \mathfrak{P}_{l}(t) & \leq \beta_{+} \frac{t^{2 H\left(n_{\left.l^{m}-2\right)}\right.}}{(2 H)^{n_{l} m-2}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left(t-s_{m}\right)^{2 H} \mathrm{~d} s_{m} \prod_{l=1, \neq l^{m}}^{k} \frac{\beta_{l} t^{2 H\left(n_{l}-1+H_{+}\right)}}{(2 H)^{n_{l}-1}} \\
& =\left(\beta_{+} \frac{t^{2 H(n-k-1)+(k-1) H_{+}}}{(2 H)^{n-k-1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l}\right) \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left(t-s_{m}\right)^{2 H} \mathrm{~d} s_{m} \\
& \leq\left(\mathfrak{g}_{2 H} \beta_{+} \frac{t^{2 H(n-k-1)+(k-1) H_{+}}}{(2 H)^{n-k-1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l}\right) \Delta_{t}^{3 H+\frac{1}{2}}=: 2_{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{3 H+\frac{1}{2}}
\end{aligned}
$$

where the last inequality with the constant $\mathfrak{g}_{2 H^{-}}$which only depends on $H$ and $T$-follows from A.1, yielding Case (1).
Case (2). In this case, namely $m=j_{n_{l}}^{l}\left(=j_{n_{l} m}^{m}\right)$, for some $l \in \llbracket 1, k \rrbracket$, we again make use of 6.7 and 6.8 to rewrite the integrand in $\mathfrak{N}_{1}^{k \mathcal{D}}(m)$ in 6.6 in the form

$$
\prod_{l=1, \neq l^{m}}^{k}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(\bar{s}_{l^{m}}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l m}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l m}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{m}}\right)
$$

Therefore,

$$
\begin{aligned}
\mathfrak{N}_{1}^{k \mathcal{D}}(m) \leq & \prod_{l=1, \neq l^{m}}^{k} \int_{0}^{t}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \int_{0}^{\bar{s}_{l}} \cdots \int_{0}^{\bar{s}_{l}} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right\} \mathrm{d} \bar{s}_{l} \\
& \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \int_{0}^{s_{m}} \cdots \int_{0}^{s_{m}} \mathbf{K}\left(s_{m}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{m}}\right) K\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l m}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{m}}\right) \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l m}} \mathrm{~d} s_{m} \\
= & \prod_{l=1, \neq l^{m}}^{k} \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \Re_{l}\left(\bar{s}_{l}\right) \mathrm{d} \bar{s}_{l} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \Re_{l^{m}}\left(s_{m}\right) \mathrm{d} s_{m}=: \prod_{l=1, \neq l^{m}}^{k} \mathfrak{P}_{l}(t) \mathfrak{T}(t)
\end{aligned}
$$

Now, we compute separately the contributions of the different terms that make up the integrand. Reasoning as in the first step, namely exploiting 6.10 we write

$$
\prod_{l=1, \neq l^{m}}^{k} \mathfrak{P}_{l}(t) \leq \prod_{l=1, \neq l^{m}}^{k} \frac{\beta_{l}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}}=\frac{t^{2 H\left(n-n_{l} m-k+1\right)}+(k-1) H_{+}}{(2 H)^{n-n_{l} m-k+1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l} .
$$

Then, exploiting 6.9 , we obtain

$$
\mathfrak{T}(t)=\int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \Re_{l^{m}}\left(s_{m}\right) \mathrm{d} s_{m} \leq \frac{1}{(2 H)^{n_{l} m}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| s_{m}^{2 H\left(n_{l} m-1\right)} \mathrm{d} s_{m}
$$

The two bounds above and A. 1 yield

$$
\begin{aligned}
\mathfrak{N}_{1}^{k \mathcal{D}}(m) & \leq \frac{1}{(2 H)^{n_{l} m}}\left\{\int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| s_{m}^{2 H\left(n_{l}^{m}-1\right)} \mathrm{d} s_{m}\right\} \frac{t^{2 H\left(n-n_{l} m-k+1\right)+(k-1) H_{+}}}{(2 H)^{n-n_{l} m-k+1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l} \\
& \leq \frac{t^{2 H(n-k+1)+(k-1) H_{+}}}{(2 H)^{n-k+1}} \prod_{l=1, \neq l^{m}}^{k-1} \beta_{l} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m} \\
& \leq\left(\mathfrak{g}_{0} \frac{t^{2 H(n-k+1)+(k-1) H_{+}}}{(2 H)^{n-k+1}} \prod_{l=1, \neq l^{m}}^{k} \beta_{l}\right) \Delta_{t}^{H_{+}}=:_{1} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m) \Delta_{t}^{H_{+}}
\end{aligned}
$$

where the second inequality holds because $n_{l^{m}} \geq 1$. This concludes Case (2).
6.2. Computations for $\left|\mathfrak{C}_{2}^{n}(t)\right|$. Recall the function $\mathfrak{J}$ from the proof of Lemma 6.1 From the definition of $\mathfrak{C}_{2}^{n}(t)$ in (3.7) and the bounds for $\mathfrak{D}_{2}^{n 1}\left(\boldsymbol{s}_{n}\right)$ and $\mathfrak{D}_{2}^{n 2}\left(\boldsymbol{s}_{n}\right)$ in Proposition 3.10, we have

$$
\begin{aligned}
\left|\mathfrak{C}_{2}^{n}(t)\right|= & \left|\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \mathfrak{D}_{2}^{n 1}\left(\boldsymbol{s}_{n}\right) \mathfrak{D}_{2}^{n 2}\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}\right| \leq \int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}}\left|\mathfrak{D}_{2}^{n 1}\left(\boldsymbol{s}_{n}\right)\right|\left|\mathfrak{D}_{2}^{n 2}\left(\boldsymbol{s}_{n}\right)\right| \mathrm{d} \boldsymbol{s}_{n} \\
\leq & \int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} \overline{\mathfrak{D}_{2}^{n 1}} \overline{\mathfrak{D}_{2}^{n 2}} \mathrm{~d} \boldsymbol{s}_{n} \\
\leq & \frac{1}{n!} \int_{0}^{t} \int_{0}^{t} \cdots \int_{0}^{t}\left[\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \mathbf{K}\left(t, \boldsymbol{s}_{n}\right)+\mathbf{C}_{\psi} \mathfrak{J}\left(t, t_{i}, \boldsymbol{s}_{n}\right)\right] \overline{\mathfrak{D}_{2}^{n 2}} \mathrm{~d} \boldsymbol{s}_{n} \\
= & \frac{1}{n!}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \overline{\mathfrak{D}_{2}^{n 2}} \mathrm{~d} \boldsymbol{s}_{n}+\mathbf{C}_{\psi} \int_{[0, t]^{n}} \mathfrak{J}\left(t, t_{i}, \boldsymbol{s}_{n}\right) \overline{\mathfrak{D}_{2}^{n 2}} \mathrm{~d} \boldsymbol{s}_{n}\right\} \\
\leq & \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}}\left\{\overline{\mathbf{C}\left(t^{\gamma}-t_{i}^{\gamma}\right)} \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathfrak{b}_{j+2} 2^{j+1} \mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}}\right. \\
& \left.+\mathbf{C}_{\psi} \sum_{m=1}^{n} \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathfrak{b}_{j+2} 2^{j+1} \mathfrak{N}_{2}^{\mathcal{U} j \mathcal{D}}(m)\right\},
\end{aligned}
$$

with

$$
\begin{align*}
\mathfrak{M}_{2}^{\mathcal{U} \mathcal{D}} & :=\int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left(\int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right) \mathrm{d} \boldsymbol{s}_{n}, \\
\mathfrak{N}_{2}^{\mathcal{U}}{ }^{\mathcal{D} \mathcal{D}}(m) & :=\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right)\left(\prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right)\left(\int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right) \mathrm{d} \boldsymbol{s}_{n} . \tag{6.12}
\end{align*}
$$

We now provide bounds for $\mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}}$ and $\mathfrak{N}_{2}^{\mathcal{U} j \mathcal{D}}(m)$ :
Lemma 6.3. For any $\mathcal{U}, j, \mathcal{D}$, the following bounds hold:
where the constants are provided in Appendix $B$ and

$$
{ }_{1} \mathcal{N}_{2}:=\mathcal{U}^{c} \cup \mathcal{I}_{\mathcal{D}}^{j} \quad \text { and } \quad{ }_{2} \mathcal{N}_{2}:=\mathcal{U} \backslash \mathcal{I}_{\mathcal{D}}^{j}
$$

such that ${ }_{1} \mathcal{N}_{2} \cup{ }_{2} \mathcal{N}_{2}=\llbracket n \rrbracket$.
Armed with this lemma, we can finally prove the rest of Proposition 3.10.

Proof of Proposition 3.10 - Bound on $\left|\mathfrak{C}_{2}^{n}(t)\right|$. Exploiting the bounds for $\mathfrak{M}_{2}^{\mathcal{U}}{ }^{j \mathcal{D}}$ and $\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m)$ in Lemma 6.3 , we have (details of sum indices that are repeated from line to line are omitted)

$$
\begin{aligned}
&\left|\mathfrak{C}_{2}^{n}(t)\right| \leq \frac{\mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H^{n}}}\left\{\overline{\mathbf{C}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathfrak{b}_{j+2} 2^{j+1} \mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}}\right. \\
&\left.+\mathbf{C}_{\psi} \sum_{m=1}^{n} \sum_{\mathcal{U} \subset \llbracket n \rrbracket} \sum_{j=1}^{|\mathcal{U}|+1} \sum_{\mathcal{D} \in \mathcal{I}_{j}^{|\mathcal{U}|+1}\left(\boldsymbol{s}_{\mathfrak{o}(\mathcal{U})}, \xi\right)} \mathfrak{b}_{j+2} 2^{j+1} \mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m)\right\}, \\
& \leq \frac{\overline{\mathbf{C}} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}}\left(t^{\gamma}-t_{i}^{\gamma}\right) \sum_{\mathcal{U}, j, \mathcal{D}} \mathfrak{b}_{j+2} 2^{j+1} \overline{\mathfrak{M}_{2}^{\mathcal{U j D}}}+\frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}}[ \\
& \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in \in_{1} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}{ }_{1} \overline{\mathfrak{N}}_{2}^{\mathcal{U} \mathcal{D}} \\
&\left.(m) \Delta_{t}^{H_{+}}+\sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in \in_{2} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}{ }_{2} \overline{\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}}(m) \Delta_{t}^{3 H+\frac{1}{2}}\right]
\end{aligned}
$$

Notice that in Case (2), $m \in \mathcal{I}_{\mathcal{D}}^{l}, l \neq j$ hence $j \geq 2$. Now, similarly to the computations for ${ }_{1} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m)$ and ${ }_{2} \overline{\mathfrak{N}_{1}^{k \mathcal{D}}}(m)$ on Page 38, we have

$$
\begin{align*}
\frac{\overline{\mathbf{C}} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}} \sum_{\mathcal{U}, j, \mathcal{D}} \mathfrak{b}_{j+2} 2^{j+1} \overline{\mathfrak{M}_{2}^{\mathcal{U} \mathcal{D}}} & \leq \frac{\overline{\mathbf{C}} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}} \sum_{\mathcal{U}, j, \mathcal{D}} \mathfrak{b}_{j+2} 2^{j+1}\left|\frac{(T-t)^{H_{+}+t^{2 H(n-j+1)+H_{+}(j-1)}}}{H_{+}(2 H)^{n-j+1}} \prod_{l=1}^{j-1} \beta_{l}^{U \mathcal{D}}\right| \\
& \leq \frac{\overline{\mathbf{C}} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}} \frac{(T-t)^{H+} T^{\left(2 H+\frac{1}{2}\right) n}}{H_{+} 2 H} \max _{l=1, \cdots, j-1}\left|\beta_{l}^{\mathcal{D}}\right|^{n} 2^{n} \sum_{j=1}^{n+1} \mathfrak{b}_{j+2} 2^{j+1} S(n+1, j)=: \overline{\overline{\mathfrak{M}_{2}^{n}}}, \tag{6.13}
\end{align*}
$$

where we used that $S(\cdot, \cdot)$ is increasing in its first argument and $\sum_{\mathcal{U} \in \llbracket n \rrbracket}=2^{n}$. Furthermore,

$$
\begin{align*}
& \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in \in_{1} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}{ }_{1} \overline{\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}}(m) \\
& \leq \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}} \frac{(T-t)^{2 H}}{2 H} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in_{1} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}\left\{\left|\mathfrak{g}_{0} \frac{t^{2 H(n-j)+(j-1) H_{+}}}{(2 H)^{n-j}} \prod_{l=1}^{j} \beta_{l}^{\mathcal{U} \mathcal{D}}\right|+\left|\mathfrak{g}_{0} \frac{t^{2 H(n-j+2)+(j-2) H_{+}}}{(2 H)^{n-j+2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}\right|\right\} \\
& \leq \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H}^{n}} \frac{(T-t)^{2 H}}{2 H}\left(1 \vee T^{2 H n+\frac{n}{2}}\right) \mathfrak{g}_{0} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in \mathcal{N}_{1} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}\left\{\left|\prod_{l=1}^{j} \beta_{l}^{\mathcal{U D}}\right|+\left|\prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}\right|\right\} \\
& \leq \frac{2 \mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{1 \vee T^{H(3 n+2)+\frac{n}{2}}(2 H)^{\frac{3 n}{2}+1}}{\mathfrak{g}} \sum_{0} \sum_{\mathcal{U}, j, \mathcal{D}} \mathfrak{b}_{m \in \mathcal{N}_{1} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}\left|\bar{\beta}^{j, \mathcal{U D}}\right| \\
& \leq \frac{2 \mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{1 \vee T^{H(3 n+2)+\frac{n}{2}}(2 H)^{\frac{3 n}{2}+1}}{\mathfrak{g}} \max _{j}\left\{\left|\bar{\beta}^{j, \mathcal{U D}}\right|\right\} n 2^{n} \sum_{j=1}^{n+1} \mathfrak{b}_{j+2} 2^{j+1} S(n+1, j)=: 1 \overline{\overline{\mathfrak{N}_{2}^{n}}}, \tag{6.14}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{T^{H n}}{\sqrt{2 H^{n}}} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}{ }_{2} \overline{\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}}(m) \\
& \leq \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{1 \vee T^{(n+1) H+\frac{1}{2}}}{(2 H)^{\frac{n}{2}} H_{+}} \mathfrak{g}_{2 H} \beta_{+} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1} \frac{t^{2 H(n-j-2)+(j-2) H_{+}}}{(2 H)^{n-j-2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}} \\
& \leq \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{1 \vee T^{\left(3 H+\frac{1}{2}\right)(n-1)}}{H_{+}(2 H)^{\frac{3 n}{2}}} \mathfrak{g}_{2 H} \beta_{+} \max _{j}\left\{\left|\bar{\beta}^{j, \mathcal{U D}}\right|\right\} \sum_{\mathcal{U}, j, \mathcal{D}} \sum_{m \in_{2} \mathcal{N}_{2}} \mathfrak{b}_{j+2} 2^{j+1}, \\
& \leq \frac{\mathbf{C}_{\psi} \mathbf{C}_{\phi, \psi}}{n!} \frac{1 \vee T^{\left(3 H+\frac{1}{2}\right)(n-1)}}{H_{+}(2 H)^{\frac{3 n}{2}}} \mathfrak{g}_{2 H} \beta_{+} \max _{j}\left\{\left|\bar{\beta}^{j, \mathcal{U D}}\right|\right\} n 2^{n} \sum_{j=1}^{n+1} \mathfrak{b}_{j+2} 2^{j+1} S(n+1, j)=: 2_{2}^{\overline{\mathfrak{N}_{2}^{n}}} \tag{6.15}
\end{align*}
$$

where we again used that $\mathrm{B}\left(x, H_{+}\right)$is uniformly bounded by $\bar{\beta}>0$ and that $S(n, k)$ is increasing in its first argument.
6.2.1. Proof of Lemma 6.3 - Bound for $\mathfrak{M}_{2}^{\mathcal{U} \mathcal{D}}$. Again, to integrate the kernels we start by splitting $\llbracket n \rrbracket$ in three sets:

$$
\mathbf{K}\left(t, \boldsymbol{s}_{n}\right)=\left\{\prod_{l=1}^{j-1} K\left(t, \bar{s}_{l}\right)\right\}\left\{\prod_{l=1}^{j-1} \mathbf{K}\left(t, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \mathbf{K}\left(t, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right)
$$

This entails the representation

$$
\begin{aligned}
\mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}} & =\int_{[0, t]^{n}} \mathbf{K}\left(t, \boldsymbol{s}_{n}\right) \prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left(\int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right) \mathrm{d} \boldsymbol{s}_{n} \\
& =\int_{[0, t]^{n}} \prod_{l=1}^{j-1} K\left(t, \bar{s}_{l}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left(\int_{t}^{T} \mathbf{K}\left(t, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right) \mathrm{d} \boldsymbol{s}_{n} \\
& =\prod_{l=1}^{j-1}\left[\int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left(\prod_{k \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} \int_{0}^{t} K\left(t, s_{k}\right) K\left(\bar{s}_{l}, s_{k}\right) \mathrm{d} s_{k}\right) \mathrm{d} \bar{s}_{l}\right] \int_{t}^{T} \prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\left[\int_{0}^{t} K\left(t, s_{k}\right) K\left(r, s_{k}\right) \mathrm{d} s_{k}\right] K(r, t) \mathrm{d} r .
\end{aligned}
$$

Noticing that $K\left(t, s_{k}\right) \leq K\left(\bar{s}_{l}, s_{k}\right)$ for $s_{k} \in\left[0, \bar{s}_{l}\right], K\left(\bar{s}_{l}, s_{k}\right)=0$ for $s_{k}>\bar{s}_{l}$ and $K\left(r, s_{k}\right) \leq K\left(t, s_{k}\right)$ for $s_{k} \in[0, t]$, since $r \in[t, T]$,

$$
\begin{aligned}
\int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left(\prod_{k \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} \int_{0}^{t} K\left(t, s_{k}\right) K\left(\bar{s}_{l}, s_{k}\right) \mathrm{d} s_{k}\right) \mathrm{d} \bar{s}_{l} & \leq \int_{0}^{t} K\left(t, \bar{s}_{l}\right) \prod_{k \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} \int_{0}^{\bar{s}_{l}} K\left(\bar{s}_{l}, s_{k}\right)^{2} \mathrm{~d} s_{k} \mathrm{~d} \bar{s}_{l} \\
& =\int_{0}^{t} K\left(t, \bar{s}_{l}\right)\left(\frac{\bar{s}_{l}^{2 H}}{2 H}\right)^{n_{l}-1} \mathrm{~d} \bar{s}_{l}=\frac{\beta_{l}^{\mathcal{U}} t^{2 H\left(n_{l}-1\right)+H_{+}}}{(2 H)^{n_{l}-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{t}^{T}\left(\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} \int_{0}^{t} K\left(t, s_{k}\right) K\left(r, s_{k}\right) \mathrm{d} s_{k}\right) K(r, t) \mathrm{d} r \leq \int_{t}^{T}\left(\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} \int_{0}^{t} K\left(t, s_{k}\right)^{2} \mathrm{~d} s_{k}\right) K(r, t) \mathrm{d} r \\
&=\left|\frac{t^{2 H}}{2 H}\right|^{n_{j}^{\mathcal{U}}}+(n-|\mathcal{U}|) \\
& \int_{t}^{T} K(r, t) \mathrm{d} r=\frac{\left.t^{2 H\left(n_{j}^{\mathcal{U}}\right.}+(n-|\mathcal{U}|)\right)}{(2 H)^{n_{j}^{\mathcal{D}}+(n-|\mathcal{U}|)} \frac{(T-t)^{H_{+}}}{H_{+}}} .
\end{aligned}
$$

Finally, using $n-|\mathcal{U}| \geq 0$, this leads to

$$
\mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}} \leq \prod_{l=1}^{j-1} \frac{\beta_{l}^{\mathcal{U} \mathcal{D}} t^{2 H\left(n_{l}-1\right)+H_{+}}}{(2 H)^{n_{l}-1}} \frac{\left.t^{2 H\left(n_{j}^{\mathcal{U}}\right.}+(n-|\mathcal{U}|)\right)}{(2 H)^{n_{j}^{\mathcal{U}}}+(n-|\mathcal{U}|)} \frac{(T-t)^{H_{+}}}{H_{+}}=\frac{(T-t)^{H_{+}} t^{2 H(n-j+1)+H_{+}(j-1)}}{H_{+}(2 H)^{n-j+1}} \prod_{l=1}^{j-1} \beta_{l}^{\mathcal{U} \mathcal{D}}=: \overline{\mathfrak{M}_{2}^{\mathcal{U} j \mathcal{D}}}
$$

6.2.2. Proof of Lemma 6.3 - Bound for $\mathfrak{N}_{2}^{\mathcal{U}}{ }^{\mathcal{D}}$. First of all, let us mention again, as pointed out at the beginning of Section 5.4 that the Malliavin operators $\mathcal{D}$ correspond here to partitions of the set $\mathcal{U}$ and, in particular, that $\mathcal{I}$ are now partitions of $\mathcal{U}$, not $\llbracket n \rrbracket$, themselves. We study separately three cases, over which we sum:
(1) $m \in{ }_{1} \mathcal{N}_{2}=\mathcal{U}^{c} \cup \mathcal{I}_{\mathcal{D}}^{j}$;
(2) $m \in{ }_{2} \mathcal{N}_{2}=\mathcal{U} \backslash \mathcal{I}_{\mathcal{D}}^{j}=\bigcup_{l=1}^{j-1} \mathcal{I}_{\mathcal{D}}^{l}$ :
(a) $m \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}$, with $l \in \llbracket 1, j-1 \rrbracket$;
(b) $m=j_{n_{l}^{\mathcal{D}}}^{l}$ for some $l \in \llbracket 1, j-1 \rrbracket$, namely $m=j_{n_{l} m}^{l^{m}}$.

Cases (1) and (2b) provide a contribution of order $H_{+}$, whereas Case (2a) provides a contribution of order $\left(3 H+\frac{1}{2}\right)$. First recall the definition of $\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m)$ 6.12;:
$\mathfrak{N}_{2}^{\mathcal{U}}{ }^{\mathcal{D} \mathcal{D}}(m)=\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right)\left(\prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right)\left(\int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right) \mathrm{d} \boldsymbol{s}_{n}$.
Compared to Section 6.1 .2 there is an additional set but the proofs are similar in spirit: we look to separate the kernels according to their second argument and isolate $s_{m}$.

Case (1) In this case, we let $\Upsilon:={ }_{1} \mathcal{N}_{2} \backslash\{m\}$ and split the kernels according to their second argument

$$
\mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right)=\left\{\prod_{l=1}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right)\right\}\left\{\prod_{l=1}^{j-1} \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right)
$$

Note also that $\mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{M}^{\mathcal{c}}}\right)=K\left(r, s_{m}\right) K\left(r, \boldsymbol{s}_{\Upsilon}\right)$ so it is convenient to rewrite the integrand in $\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m)$ in 6.12) as

$$
\left\{\prod_{l=1}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}\left\{\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right) \int_{t}^{T} K\left(r, s_{m}\right) \mathbf{K}\left(r, \boldsymbol{s}_{\Upsilon}\right) K(r, t) \mathrm{d} r\right\}
$$

where $\mathfrak{m}\left(s_{j}\right)=t_{i}$, if $j<m$, and $\mathfrak{m}\left(s_{j}\right)=t$, if $j>m$ and. Exploiting the aforementioned inequalities $K\left(r, s_{k}\right) \leq K\left(t, s_{k}\right)$ for $s_{k} \in[0, t]$, since $r \in[t, T]$ and $K\left(r, s_{m}\right) \leq K(r, t)$ on the same time interval, we have, for the term inside $\{\cdots\}$,

$$
\begin{aligned}
& \int_{t}^{T} K(r, t)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(r, s_{m}\right) \mathbf{K}\left(r, \boldsymbol{s}_{\Upsilon}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right) \mathrm{d} r \\
& \leq \mathbf{K}\left(t, \boldsymbol{s}_{\Upsilon}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \int_{t}^{T} K(r, t)^{2} \mathrm{~d} r=\frac{(T-t)^{2 H}}{2 H} \mathbf{K}\left(t, \boldsymbol{s}_{\Upsilon}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m) \leq & \int_{[0, t]^{n}} \prod_{l=1}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\{\cdots\} \mathrm{d} \boldsymbol{s}_{n} \\
\leq & \int_{[0, t]^{n}} \prod_{l=1}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left\{\frac{(T-t)^{2 H}}{2 H} \mathbf{K}\left(t, \boldsymbol{s}_{\Upsilon}\right) \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\Upsilon}\right), \boldsymbol{s}_{\Upsilon}\right)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\right\} \mathrm{d} \boldsymbol{s}_{n} \\
= & \frac{(T-t)^{2 H}}{2 H} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m} \prod_{k \in \Upsilon} \int_{0}^{t} K\left(t, s_{k}\right) K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right) \mathrm{d} s_{k} . \\
& \cdot \prod_{l=1}^{j-1} \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right)\left(\prod_{k \in \widetilde{\tilde{\mathcal{I}}}_{\mathcal{D}}^{l}} \int_{0}^{t} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right) K\left(\bar{s}_{l}, s_{k}\right) \mathrm{d} s_{k}\right) \mathrm{d} \bar{s}_{l} \\
= & \frac{(T-t)^{2 H}}{2 H} \mathfrak{R}(t)^{\left|\mathcal{U}^{c}\right|+n_{j}^{U \mathcal{D}}-1} \prod_{l=1}^{j-1} \mathfrak{P}_{l}(t)\left(\int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m}\right) .
\end{aligned}
$$

Combining the inequalities in (6.9)- (6.10) with A.1) (with $\alpha=0$ ) yields

$$
\begin{aligned}
& \mathfrak{N}_{2}^{\mathcal{U}}{ }^{\mathcal{D}}(m)=\frac{(T-t)^{2 H}}{2 H} \mathfrak{R}(t)^{\left|\mathcal{U}^{c}\right|+n_{j}^{\mathcal{U}}}-1 \\
& \prod_{l=1}^{j-1} \mathfrak{P}_{l}(t)\left(\int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m}\right) \\
& \leq \frac{(T-t)^{2 H}}{2 H}\left(\frac{t^{2 H}}{2 H}\right)^{\left|\mathcal{U}^{c}\right|+n_{j}^{\mathcal{D}}}-1 \\
& \prod_{l=1}^{j-1} \frac{\beta_{l}^{\mathcal{D} \mathcal{D}}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m} \\
&=\frac{(T-t)^{2 H}}{2 H}\left|\frac{t^{2 H(n-j)+(j-1) H_{+}}}{(2 H)^{n-j}} \prod_{l=1}^{j} \beta_{l}^{\mathcal{U D}}\right| \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m} \\
& \leq \frac{(T-t)^{2 H}}{2 H}\left|\mathfrak{g}_{0} \frac{t^{2 H(n-j)+(j-1) H_{+}}}{(2 H)^{n-j}} \prod_{l=1}^{j} \beta_{l}^{\mathcal{U D}}\right| \Delta_{t}^{H_{+}=: \overline{\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}} \Delta_{t}^{H_{+}} .} .
\end{aligned}
$$

since $\left|\mathcal{U}^{c}\right|+\sum_{l=1}^{j} n_{l}=\left|\mathcal{U}^{c}\right|+|\mathcal{U}|=n$, as $\mathcal{U}$ is a subset of $\llbracket n \rrbracket$ and $\left(n_{l}\right)_{l \in \llbracket j \rrbracket}$ represent the cardinalities of a partition of the set $\mathcal{U}$ itself.

Case (2a) We start with a general remark that holds for both sub-cases (2a) and (2b): by the inequality $\mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}}\right) \leq$ $\mathbf{K}\left(t, s_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right)$ for all $r \in[t, T]$, we have

$$
\begin{align*}
\mathfrak{N}_{2}^{\mathcal{U} \mathcal{D}}(m) & =\int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, \boldsymbol{s}_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, \boldsymbol{s}_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\left|\int_{t}^{T} \mathbf{K}\left(r, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) K(r, t) \mathrm{d} r\right| \mathrm{d} \boldsymbol{s}_{n} \\
& \leq \frac{(T-t)^{H_{+}}}{H_{+}} \int_{[0, t]^{n}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right) \prod_{l=1}^{j-1} \mathbf{K}\left(\bar{s}_{l}, \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right) \mathbf{K}\left(t, \boldsymbol{s}_{\mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}}\right) \mathrm{d} \boldsymbol{s}_{n} . \tag{6.16}
\end{align*}
$$

We now play a game similar to Section 6.1.2 we split the kernels into three groups depending on the second argument: (i) $s_{i}$ with $i \in \mathcal{I}_{\mathcal{D}}^{l}$ with $l \neq l^{m}$, (ii) $s_{i}$ with $i \in \mathcal{I}_{\mathcal{D}}^{l}$ with $l=l^{m}$, (iii) $s_{i}$ with $i \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}$. Furthermore, we separate the terms where $s_{i} \in \mathcal{I}_{\mathcal{D}}^{l}$ and $s_{i}=\bar{s}_{l}$.

$$
\mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right)=\left\{\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right)\right\}\left\{\prod_{l=1 \neq l^{m}}^{j-1} \mathbf{K}\left(\mathfrak{m}\left(s_{\mathcal{I}_{\mathcal{D}}^{l}}\right), s_{\mathcal{I}_{\mathcal{D}}^{l}}\right)\right\}\left\{\prod_{i \in \mathcal{I}^{l m} \backslash m} K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right\}
$$

$$
\begin{aligned}
=\{ & \left.\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right)\right\}\left\{\prod_{l=1 \neq l^{m}}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \mathbf{K}\left(\mathfrak{m}\left(s_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\} \\
& \cdot\left\{K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right) \prod_{i \in \widetilde{\mathcal{I}}^{l^{m}} \backslash m} K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right\}
\end{aligned}
$$

It is convenient to bound the integrand in 6.16 with the following expression:

$$
\begin{aligned}
& \left\{\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right)\right\} K\left(t, s_{k}\right)\left\{\prod_{l=1, \neq l^{m}}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\left[K\left(\bar{s}_{l}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right]\right\} \\
& K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right)\left\{\prod_{i \neq m \in \widetilde{\mathcal{I}}^{l^{m}}}\left[K\left(\bar{s}_{l^{m}}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right]\right\}
\end{aligned}
$$

Note that $m \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l^{m}}$ hence $n_{l^{m}} \geq 2$, but if it is equal to 2 then the last product is empty. We can separate the integrals

$$
\begin{aligned}
\mathfrak{N}_{2}^{\mathcal{U} j \mathcal{D}}(m) \leq & \frac{(T-t)^{H_{+}}}{H_{+}} \prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} \int_{0}^{t} K\left(t, s_{k}\right) K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right) \mathrm{d} s_{k} \\
& \prod_{l=1, \neq l^{m}}^{j-1} \int_{0}^{t}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \int_{\left[0, \bar{s}_{l}\right]^{n_{l}-1}} \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\left[K\left(\bar{s}_{l}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right] \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right\} \mathrm{d} \bar{s}_{l} \\
& \int_{0}^{t} K\left(\mathfrak{m}\left(\bar{s}_{l^{m}}\right), \bar{s}_{l^{m}}\right)\left\{\int_{0}^{\bar{s}_{l^{m}}}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| K\left(\bar{s}_{l^{m}}, s_{m}\right) \mathrm{d} s_{m}\right. \\
& \left.\int_{\left[0, \bar{s}_{l} m\right]^{n} l^{m}-2} \prod_{i \neq m \in \widetilde{\mathcal{I}}^{m}}\left[K\left(\bar{s}_{l^{m}}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right] \mathrm{d} \boldsymbol{s}_{\tilde{\mathcal{I}}^{m}} \backslash\{m\}\right\} \mathrm{d} \bar{s}_{l^{m}} \\
= & \frac{(T-t)^{H_{+}}}{H_{+}} \mathfrak{R}(t)^{n_{j}^{\mathcal{U} \mathcal{D}}+\left|\mathcal{U}^{c}\right| \mathfrak{Q}(t) \prod_{l=1, \neq l^{m}}^{j-1} \mathfrak{P}_{l}(t) .}
\end{aligned}
$$

Applying the inequalities in 6.9, 6.10 and 6.11, we obtain

$$
\begin{aligned}
& \mathfrak{N}_{2}^{\mathcal{U}}{ }^{\mathcal{D} \mathcal{D}} \\
&(m) \leq \frac{(T-t)^{H_{+}}}{H_{+}}\left(\frac{t^{2 H}}{2 H}\right)^{n-|\mathcal{U}|+n_{j}^{\mathcal{U} \mathcal{D}}} \beta_{+} \frac{t^{2 H\left(n_{l} m-2\right)}}{(2 H)^{n_{l} m-2}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left(t-s_{m}\right)^{2 H} \mathrm{~d} s_{m} \\
& \prod_{l=1, \neq l^{m}}^{j-1} \frac{\beta_{l}^{\mathcal{U D}}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1+H_{+}\right)} \\
&=\frac{(T-t)^{H_{+}}}{H_{+}} \beta_{+}\left\{\frac{t^{2 H(n-j-2)+(j-2) H_{+}}}{(2 H)^{n-j-2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}\right\} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left(t-s_{m}\right)^{2 H} \mathrm{~d} s_{m} \\
& \leq \frac{(T-t)^{H_{+}}}{H_{+}}\left(\mathfrak{g}_{2 H} \beta_{+} \frac{t^{2 H(n-j-2)+(j-2) H_{+}}}{(2 H)^{n-j-2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}\right) \Delta_{t}^{3 H+\frac{1}{2}}
\end{aligned}
$$

where in the last inequality we used A.1, concluding the first sub-case.
Case (2b) We again want to separate the kernels. Excluding $m$ is easier than in Case (2a) because we fixed it $\left(m=j_{n_{l}^{\mathcal{D}}}^{l} \notin \widetilde{\mathcal{I}}_{\mathcal{D}}^{m}\right)$, this yields
$\mathbf{K}\left(t, s_{\llbracket m+1, n \rrbracket}\right) \mathbf{K}\left(t_{i}, s_{\llbracket 1, m-1 \rrbracket}\right)=\left\{\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right)\right\}\left\{\prod_{l=1 \neq l^{m}}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right)\right\}\left\{\prod_{l=1 \neq l^{m}}^{j-1} \mathbf{K}\left(\mathfrak{m}\left(\boldsymbol{s}_{\tilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right), \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right)\right\}$.
We can thus write the integrand in 6.16 as follows:

$$
\begin{equation*}
\left\{\prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right)\right\} K\left(t, s_{k}\right)\left\{\prod_{l=1, \neq l^{m}}^{j-1} K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}} K\left(\bar{s}_{l}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right\} \tag{6.17}
\end{equation*}
$$

$$
\left|\Delta K\left(t, t_{i}, s_{m}\right)\right|\left\{\prod_{i \in \widetilde{\mathcal{I}}^{l^{m}}} K\left(\bar{s}_{l^{m}}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right\} .
$$

Thus, we obtain

$$
\begin{aligned}
\mathfrak{N}_{2}^{\mathcal{U} j \mathcal{D}}(m) \leq & \frac{(T-t)^{H_{+}}}{H_{+}} \prod_{k \in \mathcal{I}_{\mathcal{D}}^{j} \cup \mathcal{U}^{c}} \int_{0}^{t} K\left(t, s_{k}\right) K\left(\mathfrak{m}\left(s_{k}\right), s_{k}\right) \mathrm{d} s_{k} \\
& \prod_{l=1, \neq l^{m}}^{j-1} \int_{0}^{t}\left\{K\left(\mathfrak{m}\left(\bar{s}_{l}\right), \bar{s}_{l}\right) \int_{\left[0, \bar{s}_{l}\right]^{n_{l}-1}} \prod_{i \in \widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\left[K\left(\bar{s}_{l}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right] \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l}}\right\} \mathrm{d} \bar{s}_{l} \\
& \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \int_{\left[0, s_{m}\right]^{n_{l} m-1}} \prod_{i \in \widetilde{\mathcal{I}}^{m}}\left[K\left(s_{m}, s_{i}\right) K\left(\mathfrak{m}\left(s_{i}\right), s_{i}\right)\right] \mathrm{d} \boldsymbol{s}_{\widetilde{\mathcal{I}}_{\mathcal{D}}^{l^{m}}} \mathrm{~d} s_{m} \\
= & \frac{(T-t)^{H_{+}}}{H_{+}} \mathfrak{R}(t)^{n_{j}^{\mathcal{U}}+n-|\mathcal{U}|} \prod_{l=1, \neq l^{m}}^{j-1} \mathfrak{P}_{l}(t) \mathfrak{T}(t)
\end{aligned}
$$

Now, we compute separately the contributions of the different terms that make up the integrand. Reasoning as in the first step, namely exploiting 6.10 we write

$$
\prod_{l=1, \neq l^{m}}^{j-1} \mathfrak{P}_{l}(t) \leq \prod_{l=1, \neq l^{m}}^{j-1} \frac{\beta_{l}^{\mathcal{U D}}}{(2 H)^{n_{l}-1}} t^{2 H\left(n_{l}-1\right)+H_{+}}=\frac{t^{2 H\left(|\mathcal{U}|-n_{l} m-n_{j}^{\mathcal{U}}-j+2+(j-2) H_{+}\right.}}{(2 H)^{|\mathcal{U}|-n_{l} m-n_{j}^{\mathcal{U}}-j+1}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}
$$

Then, exploiting 6.9, we obtain

$$
\mathfrak{T}(t)=\int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \Re_{l^{m}}\left(s_{m}\right) \mathrm{d} s_{m} \leq \frac{1}{(2 H)^{n_{l}^{m}}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| s_{m}^{2 H\left(n_{l}^{m}-1\right)} \mathrm{d} s_{m}
$$

Finally, another application of the two bounds we have found above, together with inequality A.1) (with $\alpha=0$ ) yields $\mathfrak{N}_{2}^{\mathcal{U}{ }^{\mathcal{D}}}(m)$
$\leq \frac{(T-t)^{H_{+}}}{H_{+}}\left|\frac{t^{2 H}}{2 H}\right|^{n-|\mathcal{U}|+n_{j}^{\mathcal{U} \mathcal{D}}} \frac{1}{(2 H)^{n_{l} m}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| s_{m}^{2 H\left(n_{l} m-1\right)} \mathrm{d} s_{m} \frac{t^{2 H\left(|\mathcal{U}|-n_{j}^{\mathcal{U} \mathcal{D}}-n_{l} m-j+2\right)+(j-2) H_{+}}}{(2 H)^{|\mathcal{U}|-n_{j}^{\mathcal{U} \mathcal{D}}-n_{l}^{m}-j+2}} \prod_{l=1, \neq l^{m}}^{j} \beta_{l}^{\mathcal{U} \mathcal{D}}$
$=\frac{(T-t)^{H_{+}}}{H_{+}} \frac{t^{2 H\left(n-n_{l} m-j+2\right)+(j-2) H_{+}}}{(2 H)^{n-j+2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| s_{m}^{2 H\left(n_{l} m-1\right)} \mathrm{d} s_{m}$
$\leq \frac{(T-t)^{H_{+}}}{H_{+}} \frac{t^{2 H(n-j+2)+(j-2) H_{+}}}{(2 H)^{n-j+2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}} \int_{0}^{t}\left|\Delta K\left(t, t_{i}, s_{m}\right)\right| \mathrm{d} s_{m}$
$\leq \frac{(T-t)^{H_{+}}}{H_{+}}\left|\mathfrak{g}_{0} \frac{t^{2 H(n-j+2)+(j-2) H_{+}}}{(2 H)^{n-j+2}} \prod_{l=1, \neq l^{m}}^{j-1} \beta_{l}^{\mathcal{U D}}\right| \Delta_{t}^{H_{+}}:={ }_{2} \overline{\mathfrak{N}_{2}^{\mathcal{U} j \mathcal{D}}} \Delta_{t}^{H_{+}}$.
Summing the bounds obtained in 6.17 and 6.18, and bounding $\Delta_{t}^{3 H+\frac{1}{2}}$ by $\Delta_{t}^{H_{+}}$we obtain $\overline{\mathfrak{N}}_{2}^{\mathcal{U} j \mathcal{D}}(m) \Delta_{t}^{H_{+}}$.

## 7. Proof of Proposition 3.11 - Convergence of the series

We recall from Equation (3.7) and Proposition 3.10 that

$$
\begin{align*}
& \left|\mathfrak{B}_{1}(t)\right|=\left|\sum_{n=0}^{\infty} \mathfrak{C}_{1}^{n}(t)\right| \leq \sum_{n=0}^{\infty}\left\{\left(t^{\gamma}-t_{i}^{\gamma}\right) \overline{\overline{\mathfrak{M}_{1}^{n}}}+{ }_{1} \overline{\overline{\mathfrak{N}_{1}^{n}}} \Delta_{t}^{H_{+}}+{ }_{2} \overline{\overline{\mathfrak{N}_{1}^{n}}} \Delta_{t}^{3 H+\frac{1}{2}}\right\} \\
& \left|\mathfrak{B}_{2}(t)\right|=\left|\sum_{n=0}^{\infty} \mathfrak{C}_{2}^{n}(t)\right| \leq \sum_{n=0}^{\infty}\left\{\left(t^{\gamma}-t_{i}^{\gamma}\right) \overline{\overline{\mathfrak{M}_{2}^{n}}}+{ }_{1} \overline{\overline{\mathfrak{N}_{2}^{n}}} \Delta_{t}^{H_{+}}+{ }_{2} \overline{\overline{\mathfrak{N}_{2}^{n}}} \Delta_{t}^{3 H+\frac{1}{2}}\right\} \tag{7.1}
\end{align*}
$$

From the definitions of the constants in Appendix B we observe that there exists a constant $C=$ $C(H, T, \phi, \psi)$ such that

$$
\begin{aligned}
& \overline{\overline{\mathcal{M}_{1}^{n}}}+{ }_{1} \overline{\overline{\mathfrak{N}_{1}^{n}}}+{ }_{2} \overline{\overline{\mathfrak{N}_{1}^{n}}} \leq C^{n} \sum_{k=1}^{n} \mathfrak{b}_{k} S(n, k) \\
& \overline{\overline{\mathcal{M}_{2}^{n}}}+{ }_{1} \overline{\overline{\mathcal{N}_{2}^{n}}}+{ }_{2} \overline{\overline{\mathfrak{N}_{2}^{n}}} \leq C^{n} \sum_{j=1}^{n+1} \mathfrak{b}_{j+2} S(n+1, j)
\end{aligned}
$$

Hence, by Proposition 3.10 the series 7.1 converge provided that

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{C^{n}}{n!} \mathfrak{b}_{k+3} S(n, k)<\infty \tag{7.2}
\end{equation*}
$$

Recall [25, Eq (10)] that the optimal BDG constant for continuous martingales satisfies $\mathfrak{b}_{k} \leq(4 k)^{k / 2}$ and the Stirling number is given by

$$
S(n, k)=\frac{1}{k!} \sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(k-j)^{n}
$$

Noting $4^{(k+3) / 2} \leq 2^{n+3}$, this allows us to bound $\sqrt[7.2]{ }$ by 8 times

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=0}^{k} \frac{(2 C(k-j))^{n}}{n!} \frac{(k+3)^{\frac{k+3}{2}}}{k!} 1^{j}\binom{k}{j} & =\sum_{k=1}^{\infty} \sum_{j=0}^{k} \frac{(k+3)^{\frac{k+3}{2}}}{k!} 1^{j}\binom{k}{j}\left(\sum_{n=k}^{\infty} \frac{2 C(k-j))^{n}}{n!}\right) \\
& \leq \sum_{k=1}^{\infty} \frac{(k+3)^{\frac{k+3}{2}}}{k!}\left(\sum_{j=0}^{k}\binom{k}{j} 1^{j}\left(\mathrm{e}^{2 C}\right)^{k-j}\right) \\
& =\sum_{k=1}^{\infty} \frac{(k+3)^{\frac{k+3}{2}}}{k!}\left(\mathrm{e}^{2 C}+1\right)^{k}=: \sum_{k=1}^{\infty} a_{k} .
\end{aligned}
$$

We brutally replaced $(-1)^{j}$ by $1^{j}$, used Fubini because all the terms are positive and bounded the sum in $n$ above by an exponential. The ratio test ensures convergence:

$$
\begin{equation*}
\frac{a_{k}}{a_{k+1}}=\frac{1}{\mathrm{e}^{2 C}+1} \frac{k+1}{\sqrt{k+4}}\left(\frac{k+3}{k+4}\right)^{\frac{k+3}{2}} \tag{7.3}
\end{equation*}
$$

With $x=k+3$, the third factor in (7.3) reads $\exp \left(\frac{x}{2} \ln \frac{x}{x+1}\right)$, and l'Hôpital's rule implies

$$
\lim _{x \uparrow \infty} x \ln x-x \ln (x+1)=\lim _{x \uparrow \infty} \frac{\ln x-\ln (x+1)}{1 / x}=\lim _{x \uparrow \infty} \frac{1 /(x(x+1))}{-1 / x^{2}}=\lim _{x \uparrow \infty}-\frac{x}{x+1}=-1
$$

Hence the third factor of 7.3 converges to $\mathrm{e}^{-\frac{1}{2}}$, so $a_{k} / a_{k+1}$ tends to infinity and the series is finite.

## Appendix A. Technical lemmas

Lemma A.1. Let $n \in \mathbb{N}$ and $f: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ be symmetric then

$$
\int_{[0, t]^{n}} f\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}=n!\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} f\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}
$$

Proof. For a function $F: \mathbb{R}^{n} \rightarrow \mathbb{R}$, define its symmetrisation $\widetilde{F}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ as

$$
\widetilde{F}\left(s_{n}\right):=\frac{1}{n!} \sum_{\sigma} F\left(s_{\sigma(1)}, \cdots, s_{\sigma(n)}\right)
$$

summing over all permutations of $\llbracket n \rrbracket$, and notice that $\int_{[0, t]^{n}} F\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}=\int_{[0, t]^{n}} \widetilde{F}\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n}$. Let $f$ : $\mathbb{R}_{+}^{n} \rightarrow \mathbb{R}$ be symmetric, then

$$
\begin{aligned}
\int_{0}^{t} \int_{0}^{s_{n}} \cdots \int_{0}^{s_{2}} f\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n} & =\int_{[0, t]^{n}} \mathbb{1}_{s_{1}<s_{2}<\cdots<s_{n}} f\left(\boldsymbol{s}_{n}\right) \mathrm{d} \boldsymbol{s}_{n} \\
& =\int_{[0, t]^{n}} \frac{1}{n!} \sum_{\sigma} \mathbb{1}_{s_{\sigma(1)}<s_{\sigma(2)}<\cdots<s_{\sigma(n)}} f\left(s_{\sigma(1)}, \cdots, s_{\sigma(n)}\right) \mathrm{d} \boldsymbol{s}_{n}
\end{aligned}
$$

By symmetry, $f\left(s_{\sigma(1)}, \cdots, s_{\sigma(n)}\right)=f\left(s_{n}\right)$ for any ordering of $s_{n}$ and $\sum_{\sigma} \mathbb{1}_{s_{\sigma(1)}<s_{\sigma(2)}<\cdots<s_{\sigma(n)}}=1$, which yields the claim.

Lemma A.2. The following (in)equalities hold for all $s, t_{i} \in \mathbb{T}, t \in\left[t_{i}, t_{i}+\Delta_{t}\right]$ and $\alpha \geq 0, \beta>-1$ :

$$
\begin{align*}
& \int_{0}^{t}\left|\Delta K\left(t, t_{i}, r\right)\right|(t-r)^{\alpha} \mathrm{d} r \leq \mathfrak{g}_{\alpha} \Delta_{t}^{-\left(\alpha+H_{+}\right)}, \quad \mathfrak{g}_{\alpha}>0  \tag{A.1}\\
& \int_{0}^{t} K(t, r) r^{\beta} \mathrm{d} r=\mathrm{B}\left(H_{+}, \beta+1\right) t^{\beta+H_{+}}  \tag{A.2}\\
& \int_{0}^{t} K(s, r) K(t, r) \mathrm{d} r \leq \frac{(s \wedge t)^{2 H}}{2 H} \tag{A.3}
\end{align*}
$$

Proof. The first inequality (A.1) is due to [18 Lemma 2.1, Eq. (2.2)]. The second one (A.2) is a standard property of Beta functions. For the proof of A.3) we recall that $K(t, r)=0$ if $r>t$ hence

$$
\int_{0}^{t} K(s, r) K(t, r) \mathrm{d} r=\int_{0}^{s} K(s, r) K(t, r) \mathrm{d} r
$$

Without loss of generality we can then assume $s \leq t$. For all $r \in[0, s], K(s, r) \geq K(t, r)$ and therefore

$$
\int_{0}^{s} K(s, r) K(t, r) \mathrm{d} r \leq \int_{0}^{s} K(s, r)^{2} \mathrm{~d} r=\frac{s^{2 H}}{2 H}
$$

## Appendix B. Table of Constants



- The constants $\mathbf{C}_{\psi}$ and $\mathbf{C}_{\psi^{2}}$ are the $L^{1}$ bound of $\psi^{(n)}\left(V_{u}\right)$ and $\psi^{2(n)}\left(V_{u}\right)$ on Page 30
- The constant $\mathbf{C}_{\phi, \psi}$ on Page 32 only depends on the regularities of $\phi, \psi$ and their derivatives;
- $\xi \in \mathbb{R}$ is a dummy variable and $\mathfrak{b}_{p}$ is the BDG constant which is bounded by $(4 p)^{p / 2}$;


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[^0]:    ${ }^{1}$ The authors would like to thank Jean-François Chassagneux for pointing out this nice method of proof.

[^1]:    ${ }^{2}$ Notice that $\bigcup_{k=1}^{n} \bigcup_{j=1}^{S(n, k)} \mathcal{I}_{k}^{n}(j)=\bigcup_{k=1}^{n}\left\{\left(j_{1}, \ldots, j_{k}\right): j_{i} \in \llbracket n \rrbracket, i \in \llbracket 1, k \rrbracket, j_{1}<j_{2}<\cdots<j_{k}\right\}$, as in Section 5.1

