

THE LOG-MOMENT FORMULA FOR IMPLIED VOLATILITY

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Dedicated to the memory of Mark H.A Davis

ABSTRACT. We revisit the foundational Moment Formula proved by Roger Lee fifteen years ago. We show that in the absence of arbitrage, if the underlying stock price at time T admits finite log-moments $\mathbb{E}[|\log S_T|^q]$ for some positive q , the arbitrage-free growth in the left wing of the implied volatility smile for T is less constrained than Lee's bound. The result is rationalised by a market trading discretely monitored variance swaps wherein the payoff is a function of squared log-returns, and requires no assumption for the underlying price to admit any negative moment. In this respect, the result can be derived from a model-independent setup. As a byproduct, we relax the moment assumptions on the stock price to provide a new proof of the notorious Gatheral-Fukasawa formula expressing variance swaps in terms of the implied volatility.

1. INTRODUCTION

Implied volatility is at the very core of Quantitative Finance and is the day-to-day gadget that traders observe and manipulate. The increasing complexity of stochastic models we have witnessed over the past thirty years is a testimony to its importance and subtlety. One key issue though is the absence of closed-form expression for the latter, leaving it to the sometimes capricious moods of numerical analysis. Among the plethora of research in this direction, carried out both by academics and by practitioners, model-free results, with minimum assumptions, are scarce. Roger Lee's Moment Formula [31] was a groundbreaking result and its importance cannot be understated: it provides a direct link between the slope of the smile in the wings and the moments of the distribution of the underlying asset price. It serves not only to infer directly observed information about the implied volatility smile into constraints on model parameters but also to provide arbitrage-free solutions to the extrapolation problem (how to evaluate options for strikes outside the observed range). Recent refinements have led to a deeper understanding of the information contained in the implied volatility smile, determining whether the probability of default of the underlying could be inferred [15, 24] or the potential lack of martingality of the latter [28]. These have complemented the otherwise exhaustive literature on the asymptotic behaviour of stochastic models in Finance, a thorough review of which can be consulted in [18].

Asymptotic methods have both supporters and enemies, the former trying to expand the abundance of techniques to every possible model, while the latter sometimes dismiss the usefulness of these results. The truth as often lies somewhere in the middle but asymptotic results nevertheless provide useful information about the qualities and pitfalls of models with regards to real-life practices. With this in mind, we revisit Lee's formula

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when presented with some underlying stock price, the prices of finitely many co-maturing European Call and Put options as well as a variance swap with the same maturity. In [13] conditions were stated under which a given set of European Call and Put prices all maturing at the same time T is consistent with absence of arbitrage, which is shown to be equivalent to the existence of a *market model*; a filtered probability space carrying an adapted, integrable process $(S_t)_{t \in [0, T]}$, with S_0 equal to the time-0 stock price, in which the discounted stock price process is a martingale and the discounted expectations of the Put option payoffs recover the observed time-zero values. In [14] robust model-free conditions are provided, when the process $(S_t)_{t \in [0, T]}$ admits continuous sample paths, for a set of European Put option prices and *continuously monitored* variance swap price to be consistent with absence of arbitrage. Our approach here is to make no assumption on the dynamics of the stock-price process, and to instead infer limiting behaviour of the left-wing given merely the information that the marginal distribution of S_T admits finite log moments, $\mathbb{E}[|\log S_T|^q]$ for finite positive values of q , which is motivated directly from the market since it trades a (discretely or continuously monitored) variance swap. Further, it is feasible for market models to exist that do not admit any negative moment for the stock price and Lee's Moment Formula (on S_T rather than $\log(S_T)$) implies that the left wing of the smile has slope precisely equal to two. Our newly formulated Log-Moment Formula allows us to provide higher-order term in this asymptotic behaviour, fully characterised by the moments of the log-stock price.

We provide a precise formulation of the problem and a thorough review of Roger Lee's Moment Formula in Section 2 before stating and proving the new Log-Moment Formula in Section 3. As a byproduct, we revisit the Fukasawa-Gatheral formula expressing variance swaps in terms of the implied volatility, provide a new proof with relaxed assumptions and further show how this improves Fukasawa's representation [19] of option prices in terms of implied volatility. We highlight in Section 4 a few stochastic models, both with continuous paths and with jumps, used in Finance for which our formula refines Lee's standards.

2. PROBLEM FORMULATION AND BACKGROUND

We consider a time horizon $[0, T]$, with $T > 0$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0, T]}, \mathbb{Q})$ satisfying the usual assumptions and carrying two adapted processes $(\bar{S}_t)_{t \in [0, T]}$, the asset price, and $(B_t)_{t \in [0, T]}$, starting from $B_0 = 1$, where B_T represents the value at time T of £1 invested at time 0 in the money-market account. We further denote by F_t the t -forward price of \bar{S} from time 0, thus $F_0 = \bar{S}_0$ is the observed spot price. Dividends may be paid by the asset \bar{S} , but we do not make any assumptions about these. The process $(\bar{S}_t)_{t \in [0, T]}$ is assumed to be a strictly positive \mathbb{Q} -semimartingale. We finally assume the existence of a zero-coupon-bond maturing at T with face-value £1, traded with price P_T at time zero such that F_T and P_T are consistent with absence of arbitrage; we further assume that the probability measure \mathbb{Q} is a T -forward neutral measure, meaning that the prices of all traded assets, expressed using the zero coupon with maturity T as numéraire, are local martingales. In particular, since bounded local martingales are true martingales, the price of a vanilla Put option with strike $K \geq 0$ is given by $P_0(K) := P_T \mathbb{E}[(K - \bar{S}_T)^+]$. Using normalised units for the stock-price $S_T := \bar{S}_T / F_T$ and log-moneyness $x := \log(K / F_T)$, the normalised price of the Put option is denoted by

$$P(x) := \frac{P_0(K)}{P_T F_T} = \mathbb{E} \left[(e^x - S_T)^+ \right].$$

Recall now the Black-Scholes formula for the European Put option:

$$(2.1) \quad P_{BS}(x, \sigma) = \begin{cases} e^x \Phi[-\mathfrak{d}(x, \sigma)] - \Phi[-\mathfrak{d}(x, \sigma) - \sigma], & \text{if } \sigma \neq 0, \\ (e^x - 1)^+, & \text{if } \sigma = 0, \end{cases}$$

where Φ denotes the Gaussian cumulative distribution function and, for $\sigma \neq 0$,

$$(2.2) \quad \mathfrak{d}(x, \sigma) := -\frac{x}{\sigma} - \frac{\sigma}{2},$$

which is nothing else than the usual d_2 or d_- from the Black-Scholes formula. Since the maturity T is fixed throughout the whole paper, we work with normalised volatility σ rather than the classical $\sigma\sqrt{T}$ notation. This has the clear advantage of avoiding cluttered statements.

Definition 2.1. For any log-moneyness $x \in \mathbb{R}$, the implied volatility $I(x) \in [0, \infty)$ is the unique non-negative solution to $P(x) = P_{BS}(x, I(x))$.

The implied volatility $I(x)$ is well defined whenever $P(x) \in [(e^x - 1)^+, e^x]$, which holds since it has been assumed that \bar{S} is strictly positive and $F_T = \mathbb{E}[\bar{S}_T]$. Our starting point is the following initial bound for the implied volatility [31, Lemma 3.3]:

Lemma 2.2. *For any $\beta > 2$, there exists $x^* < 0$ such that $I(x) < \sqrt{\beta|x|}$ for all $x < x^*$. For $\beta = 2$, the same holds if and only if $\mathbb{Q}[S_T = 0] < \frac{1}{2}$.*

In our setup, S_T is strictly positive almost surely and therefore $I(x) = \mathcal{O}(\sqrt{2|x|})$ as x tends to $-\infty$. When the number of finite inverse power moments for the stock price is known, the small-strike Moment Formula due to Lee [31] refines the above result:

Theorem 2.3 (Lee's Left Moment Formula). *Let*

$$\mathfrak{p} := \sup \{p > 0 : \mathbb{E}[S_T^{-p}] < \infty\}, \quad \text{and} \quad \beta_L := \limsup_{x \downarrow -\infty} \frac{I(x)^2}{|x|}.$$

Then $\beta_L \in [0, 2]$, $\mathfrak{p} = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}$, with $\frac{1}{0} := \infty$. Equivalently, $\beta_L = 2 - 4(\sqrt{\mathfrak{p}^2 + \mathfrak{p}} - \mathfrak{p})$, equal to 0 for $\mathfrak{p} = \infty$.

This theorem was one of the first model-free results about the relationship between the distribution of the stock price and the behaviour of the implied volatility. The limsup in Lee's result was further strengthened to a genuine limit by Benaim and Friz [2, 3], albeit with additional assumptions, and necessary and sufficient conditions to replace limsup by lim can be found in [23]. It is really a cornerstone in the implied volatility modelling literature and has provided academics and practitioners robust consistency checks for extrapolation of the smile. Lee also proved a symmetric right-wing formula, but we omit its presentation as we shall not require it here. This left-wing behaviour of the smile left two unresolved issues however: if S_T has a strictly positive mass at the origin, then Lee's expression is not able to distinguish it from a mass-less distribution with fat tails; this was tackled in [15, 24]. The second issue is that in fact no information about the moments of S_T is really available in the market, and the so-called Power options [6] are rarely traded. However, variance swaps are traded on the market and it is thus a natural question to check if Lee's celebrated Moment Formula could be refined to take into account these highly liquid derivatives.

3. VARIANCE SWAPS AND THE LOG-MOMENT FORMULA

3.1. Characterisation of variance swaps. Variance swaps are highly liquid traded derivatives on the Equity market. One can describe them as a standard swap, where, over the time horizon $[0, T]$ the floating leg is equal to the (annualised) realised variance \mathfrak{V}_T^d as measured by

$$\mathfrak{V}_T^d := \frac{252}{T} \sum_{i=1}^n \log \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)^2,$$

in which $0 = t_0 < t_1 < \dots < t_n = T$ is an equidistant partition with $t_i - t_{i-1} = T/n$ corresponding to one day for some positive integer n . The superscript ^d here refers to the fact that this definition corresponds to the so-called discretely monitored variance swap. The early advances on the hedging and pricing of the variance swap by Neuberger [38], Dupire [11], Carr and Madan [7] and Demeterfi, Derman, Kamal and Zou [17] led to the instrument being used extensively by traders to express views on future realised variance and for hedging volatility risk. These advances hinged on assuming (i) the stock price process is a continuous semimartingale with strictly positive values, (ii) the realised variance is *continuously monitored* and measured by the quadratic variation $\langle \log S \rangle_T$, and (iii) Call or Put options maturing at time T are traded for all strikes $K \in \mathbb{R}_+$. Itô's formula for continuous semimartingales applied to $-\log S_T$, then gives

$$\langle \log S \rangle_T = \int_0^T \frac{d\langle S \rangle_t}{S_t^2} = -2 \log \left(\frac{S_T}{S_0} \right) + 2 \int_0^T \frac{dS_t}{S_t}.$$

The variance swap is replicated by holding a contract paying $-\log(S_T/S_0)$ and dynamic trading in the underlying stock. Now, the log payoff $-\log S_T$ is redundant, since

$$-\log \left(\frac{S_T}{S_0} \right) = \frac{S_T - S_0}{S_0} + \int_0^{S_0} \frac{(K - S_T)^+}{K^2} dK + \int_{S_0}^\infty \frac{(S_T - K)^+}{K^2} dK,$$

i.e. it is hedged by a static position in the underlying asset, the continuum of Call and Put options, and cash. The variance swap payoff in this setup is therefore fully replicated, with no assumptions on the dynamics of the price process S , except for continuity. It thus follows that the variance swap-rate is the forward cost of the full hedging portfolio. When interest rates are zero and dividends are not paid by the underlying asset, this is

$$2 \int_0^{S_0} \frac{P_0(K)}{K^2} dK + 2 \int_{S_0}^\infty \frac{C_0(K)}{K^2} dK,$$

provided both integrals are finite, with $P_0(K)$ and $C_0(K)$ the prices of Put and Call options with strike K . The subtle impact of jumps on the prices of variance swaps was treated thoroughly by Broadie and Jain [4]. In both the discretely monitored and the continuously monitored case, the moments of the underlying stock price are not at play, but rather the moments of its logarithm, thus creating the need to refine Lee's formula to this case.

3.2. The Log-Moment Formula. Our main result is the following Log-Moment Formula, proved on Page 8:

Theorem 3.1. *Let $\mathfrak{q} := \sup \{q \geq 0 : \mathbb{E} [|\log S_T|^q] < \infty\}$ be finite. Then*

$$\liminf_{x \downarrow -\infty} \frac{\mathfrak{I}(x, \mathbf{I}(x))}{\sqrt{2 \log |x|}} = \sqrt{\mathfrak{q}}.$$

It is clear that \mathfrak{q} does not provide information about the right tail of the distribution of S_T . Since the first moment of S_T is finite, then, for any $q \geq 0$, there exists some constant $c_q \geq 1$ such that

$$\mathbb{E} [|\log S_T|^q \mathbf{1}_{\{S_T \geq c_q\}}] \leq \mathbb{E} [S_T \mathbf{1}_{\{S_T \geq c_q\}}] \leq \mathbb{E} [S_T],$$

which is finite, since S_T is strictly positive almost surely. The following corollary is immediate but makes the behaviour of the implied volatility in the left wing more explicit.

Corollary 3.2. *In the setting of Theorem 3.1, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ diverging to $-\infty$ such that*

$$\lim_{n \uparrow \infty} \frac{\mathbf{I}(x_n)}{\sqrt{2\mathfrak{q} \log(|x_n|) + 2|x_n|} - \sqrt{2\mathfrak{q} \log(|x_n|)}} = 1,$$

and, when $\mathfrak{q} > 0$, for n large enough,

$$I(x_n) = \sqrt{2|x_n|} - \sqrt{2\mathfrak{q} \log |x_n|} + \frac{\mathfrak{q} \log(|x_n|)}{\sqrt{2|x_n|}} + \mathcal{O}\left(\frac{\log(|x_n|)^2}{|x_n|^{3/2}}\right).$$

Proof. Theorem 3.1 implies the existence of a subsequence $(x_n)_{n \in \mathbb{N}}$ diverging to $-\infty$ such that

$$\lim_{n \uparrow \infty} \frac{\mathfrak{d}(x_n, I(x_n))}{\sqrt{2 \log |x_n|}} = \sqrt{\mathfrak{q}}.$$

In the case where $\mathfrak{q} = 0$, the corollary is straightforward. Consider then $\mathfrak{q} > 0$. For any $\varepsilon > 0$ small enough, there exists $N_\varepsilon \in \mathbb{N}$ such that, for any $n \geq N_\varepsilon$,

$$\sqrt{\mathfrak{q}} - \varepsilon \leq \frac{\mathfrak{d}(x_n, I(x_n))}{\sqrt{2 \log |x_n|}} = \frac{1}{\sqrt{2 \log |x_n|}} \left(-\frac{x_n}{I(x_n)} - \frac{I(x_n)}{2} \right) \leq \sqrt{\mathfrak{q}} + \varepsilon.$$

We choose N_ε large enough so that $I(x_n) > 0$ and $x_n < 0$. This is equivalent to

$$(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} \leq -\frac{x_n}{I(x_n)} - \frac{I(x_n)}{2} \leq (\sqrt{\mathfrak{q}} + \varepsilon) \sqrt{2 \log |x_n|},$$

or else, since $I(x_n) > 0$,

$$(3.1) \quad -2(\sqrt{\mathfrak{q}} + \varepsilon) \sqrt{2 \log |x_n|} I(x_n) - 2x_n \leq I(x_n)^2 \leq -2(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} I(x_n) - 2x_n,$$

The right-hand side can be written as:

$$I(x_n)^2 + 2(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} I(x_n) + 2x_n \leq 0.$$

The discriminant of this quadratic on the left reads $\Delta = 8(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n| + 8|x_n|$ and is clearly strictly positive, so that the two roots are

$$I_\pm(x_n) = \frac{-2(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} \pm \sqrt{\Delta}}{2} = -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} \pm \sqrt{2(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n| + 2|x_n|}.$$

The positive root corresponds to the $+$ sign. Arguing analogously for the left-hand side of (3.1) yields

$$-(\sqrt{\mathfrak{q}} + \varepsilon) \sqrt{2 \log |x_n|} + \sqrt{2(\sqrt{\mathfrak{q}} + \varepsilon)^2 \log |x_n| + 2|x_n|} < I(x_n) < -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} + \sqrt{2(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n| + 2|x_n|},$$

and the first equality in the corollary follows by taking ε to zero (equivalently n to infinity). Now,

$$\begin{aligned} I_+(x_n) &= -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} \pm \sqrt{2(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n| + 2|x_n|} \\ &= -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} + \sqrt{2|x_n|} \sqrt{1 + \frac{(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n|}{|x_n|}} \\ &= -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} + \sqrt{2|x_n|} \left\{ 1 + \frac{1}{2} \frac{(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n|}{|x_n|} + \mathcal{O}\left(\frac{\log(|x_n|)^2}{|x_n|}\right) \right\} \\ &= -(\sqrt{\mathfrak{q}} - \varepsilon) \sqrt{2 \log |x_n|} + \sqrt{2|x_n|} + \frac{(\sqrt{\mathfrak{q}} - \varepsilon)^2 \log |x_n|}{\sqrt{2|x_n|}} + \mathcal{O}\left(\frac{\log(|x_n|)^2}{|x_n|^{3/2}}\right), \end{aligned}$$

using a Taylor expansion in the last term. A similar analysis can be carried out for the left inequality in (3.1), and the corollary holds, taking ε to zero. \square

Benaim and Friz [2, 3] refined Lee's result, with additional assumptions, from a \liminf / \limsup statement to a genuine limit. One could investigate how this might apply here, but we defer it to a future analysis in order not to clutter our main result with extra technical assumptions. An interesting feature however is the form of the small-strike implied volatility expansion in Corollary 3.2. The slope equal to 2 of the total implied variance I^2 is trivial from Lee's result (Theorem 2.3) since \mathbf{q} finite implies $\mathbf{p} = 0$ (no negative moment of the stock price exists). Lee's formulation however does not provide further details. In the case of a strictly positive mass at the origin, De Marco, Hillairet and Jacquier [15, Theorem 3.6] proved that

$$I(x) = \sqrt{2|x|} + \mathbf{c} + \varphi(x), \quad \text{as } x \downarrow -\infty,$$

where the constant \mathbf{c} is related to the mass at zero and the function φ tends to zero as x tends to infinity, which, while capturing the slope 2, is markedly different from our new formula here. Before being able to prove the theorem, we need two lemmas providing bounds on prices of Put options and on the implied volatility.

Lemma 3.3. *Let $q \geq 0$ be such that $\mathbb{E}[|\log S_T|^q]$ is finite. Then for all $x < (q-1)\mathbf{1}_{\{q < 1\}}$,*

$$P_{BS}(x, I(x)) \leq e^x |x|^{-q} \mathbb{E}[|\log S_T|^q].$$

Proof. The case $q = 0$ is a consequence of no-arbitrage bounds for the Put option. Now consider $q > 0$. For ease of exposition only, we work in the moneyness unit $K = e^x$. The map $K \mapsto |\log(K)|^q$ is strictly convex on $\mathcal{K} := (0, e^{q-1}\mathbf{1}_{\{q < 1\}} + \mathbf{1}_{\{q \geq 1\}})$. Let now $v_q(K)$ denote the solution to the equation

$$(3.2) \quad K = v_q(K) \left(1 - \frac{1}{q} \log v_q(K) \right),$$

for $K \in \mathcal{K}$, such that $\lim_{K \downarrow 0} v_q(K) = 0$. This equation can be solved explicitly as

$$v_q(K) = \exp \{ \mathcal{W}_{-1}(-qK e^{-q}) + q \}.$$

From [12], the Lambert function \mathcal{W} solves $\mathcal{W}(z)e^{\mathcal{W}(z)} = z$ for $z \geq -\frac{1}{e}$ and is multi-valued for $z \in [-\frac{1}{e}, 0)$. The \mathcal{W}_{-1} branch is the one that satisfies $\lim_{z \uparrow 0} \mathcal{W}_{-1}(z) = -\infty$. Note further that necessarily $v_q(K) \in (0, K)$: indeed (3.2) can be re-written as $v_q(K) - K = \frac{v_q(K)}{q} \log v_q(K)$. Clearly $v_q(K) = K$ is not possible otherwise $v_q(K) \in \{0, 1\}$ but $K \notin \{0, 1\}$ by assumption. Assume that $v_q(K) > K$, then $\log v_q(K)$ needs to be strictly positive, i.e. $v_q(K) > 1$, which is not consistent with $\lim_{K \downarrow 0} v_q(K) = 0$. Thus, since $v_q(K) < K$ for small K and can never be equal to K , it is always in $(0, K)$.

For any $K \in \mathcal{K}$, the map $u \mapsto \frac{1}{q} v_q(K) |\log v_q(K)|^{1-q} \log u^q$ is decreasing and convex on $u \in (0, K)$, and $u \mapsto (K - u)^+$ is also decreasing (and linear) on this interval. They further intersect precisely at $u = v_q(K)$, where furthermore both have gradients equal to -1 , and therefore

$$(K - u)_+ \leq \frac{1}{q} v_q(K) |\log v_q(K)|^{1-q} |\log(u)|^q, \quad \text{for any } u > 0,$$

where the statement is obvious for $u \geq K$ since the right-hand side is non-negative then. Replacing u with S_T and taking expectations, we obtain

$$(3.3) \quad P_{BS}(\log(K), I(\log(K))) \leq \frac{1}{q} v_q(K) |\log v_q(K)|^{1-q} \mathbb{E}[|\log S_T|^q].$$

By construction $0 < v_q(K) < K \leq 1$, hence $|\log v_q(K)|^{-q} < |\log(K)|^{-q}$. Using $\log v_q(K) < 0$ and (3.2), it holds $\frac{1}{q} v_q(K) |\log v_q(K)| < K$. Combining these, a larger bound (than in (3.3)) for Put prices is given by

$$(3.4) \quad P_{BS}(\log(K), I(\log(K))) \leq K |\log(K)|^{-q} \mathbb{E}[|\log S_T|^q].$$

□

Remark 3.4. In the limit $x \downarrow -\infty$ (or $K \downarrow 0$), we are indifferent between the bounds in (3.4) and (3.3), because

$$(3.5) \quad \lim_{K \downarrow 0} \frac{\frac{1}{q} v_q(K) |\log v_q(K)|^{1-q}}{K |\log K|^{-q}} = 1.$$

To see this, first recall that $\lim_{K \downarrow 0} v_q(K) = 0$, then from (3.2),

$$\lim_{K \downarrow 0} \frac{K}{\frac{1}{q} v_q(K) |\log v_q(K)|} = 1.$$

Further, taking logarithm of both sides of (3.2) it follows that $\lim_{K \downarrow 0} \frac{\log(K)}{\log v_q(K)} = 1$, which implies (3.5).

Lemma 3.5. *Let $q > 0$ such that $\mathbb{E}[|\log S_T|^q]$ is finite. Then for any $p \in [0, q)$, there exists $x_p < 0$ such that*

$$I(x) < \sqrt{2|x| + 2p \log(|x|)} - \sqrt{2p \log(|x|)}, \quad \text{for all } x < x_p.$$

Remark 3.6. When $q = 0$, the condition $\mathbb{E}[|\log S_T|^q]$ finite is empty and so is the set $[0, q)$. The conclusion of the lemma nevertheless still holds with $p = 0$ and $x_p = x^*$ from Lemma 2.2.

Proof. Let $q > 0$ and define the functions $f, g : (-\infty, 1) \rightarrow \mathbb{R}$ by

$$f(x) := \sqrt{p \log(|x|)} - x \quad \text{and} \quad g(x) := \sqrt{p \log(|x|)},$$

such that $f(x)^2 - g(x)^2 = -x$. Consider then an implied volatility of the form

$$I(x) = \sqrt{2}(f(x) - g(x)), \quad \text{for } x \in \mathbb{R},$$

so that, from (2.2), we have

$$\mathfrak{d}(x, I(x)) = \sqrt{2}g(x) \quad \text{and} \quad -\mathfrak{d}(x, I(x)) - I(x) = -\sqrt{2}f(x),$$

and therefore the corresponding Put option price (2.1) is given by

$$\mathbb{P}_{\text{BS}}(x, I(x)) = e^x \Phi\left(-\sqrt{2}g(x)\right) - \Phi\left(-\sqrt{2}f(x)\right).$$

With ϕ denoting the Gaussian density, the asymptotic relationship

$$(3.6) \quad \lim_{z \uparrow \infty} \frac{z\Phi(-z)}{\phi(z)} = 1,$$

holds trivially by L'Hôpital's rule and therefore

$$\lim_{x \downarrow -\infty} \frac{\Phi(-\sqrt{2}f(x))}{e^x \Phi(-\sqrt{2}g(x))} = 0,$$

which implies

$$\lim_{x \downarrow -\infty} \frac{\mathbb{P}_{\text{BS}}(x, \sqrt{2}(f(x) - g(x)))}{e^x \Phi(-\sqrt{2}g(x))} = 1.$$

We can then deduce

$$\begin{aligned}
 \lim_{x \downarrow -\infty} \frac{e^x |x|^{-q}}{\mathbb{P}_{\text{BS}}(x, \sqrt{2}(f(x) - g(x)))} &= \lim_{x \downarrow -\infty} \frac{e^x |x|^{-q}}{e^x \Phi(-\sqrt{2}g(x))} = \lim_{x \downarrow -\infty} \frac{|x|^{-q}}{\Phi(-\sqrt{2}g(x))} \\
 &= \lim_{x \downarrow -\infty} \frac{-\sqrt{2}g(x)|x|^{-q}}{\phi(-\sqrt{2}g(x))} \\
 &= \lim_{x \downarrow -\infty} \frac{2\sqrt{\pi}g(x)|x|^{-q}}{e^{-g(x)^2}} \\
 (3.7) \qquad \qquad \qquad &= \lim_{x \downarrow -\infty} 2\sqrt{\pi}g(x)|x|^{p-q} = \begin{cases} 0, & \text{if } p < q, \\ \infty, & \text{if } p \geq q, \end{cases}
 \end{aligned}$$

and the lemma follows from Lemma 3.3 and the monotonicity of $\mathbb{P}_{\text{BS}}(\cdot, \cdot)$ in its second argument. \square

Before stating the proof of Theorem 3.1, recall the following lemma, which will be used repeatedly:

Lemma 3.7. *For any convex function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$, the identity*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_0^{x_0} (y - x)^+ \mu(dy) + \int_{x_0}^\infty (x - y)^+ \mu(dy),$$

holds for Lebesgue almost all $x, x_0 \in \mathbb{R}_+$, where $\mu = f''$ in the sense of distributions.

Proof of Theorem 3.1. Let $\zeta := \liminf_{x \downarrow -\infty} \frac{\mathfrak{d}(x, \mathbf{I}(x))}{\sqrt{2 \log |x|}}$. We first prove the theorem in the case $\mathfrak{q} > 0$. Suppose by contradiction that $\zeta < \sqrt{\mathfrak{q}}$ and let q such that $\sqrt{q} \in (\zeta, \sqrt{\mathfrak{q}})$. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $(-\infty, -1)$ with $x_n \downarrow -\infty$ such that for all n , $\mathfrak{d}(x_n, \mathbf{I}(x_n)) < \sqrt{2q \log |x_n|}$. Expanding this yields

$$0 < \sqrt{2q \log |x_n|} - \mathfrak{d}(x_n, \mathbf{I}(x_n)) = \sqrt{2q \log |x_n|} + \frac{x_n}{\mathbf{I}(x_n)} + \frac{\mathbf{I}(x_n)}{2}.$$

Multiplying both sides by $2\mathbf{I}(x_n) > 0$ yields

$$0 < 2\mathbf{I}(x_n)\sqrt{2q \log |x_n|} + 2x_n + \mathbf{I}(x_n)^2 = \left(\mathbf{I}(x_n) + \sqrt{2q \log |x_n|}\right)^2 + 2x_n - 2q \log |x_n|,$$

or equivalently,

$$2q \log |x_n| - 2x_n < \left(\mathbf{I}(x_n) + \sqrt{2q \log |x_n|}\right)^2.$$

Since the map $x \mapsto 2q \log |x| - 2x$ is strictly positive on $(-\infty, -1)$ for any $q > 0$, we can take the square root on both sides, and therefore

$$\mathbf{I}(x_n) > \sqrt{2|x_n| + 2q \log |x_n|} - \sqrt{2q \log |x_n|},$$

which contradicts Lemma 3.5 since $q < \mathfrak{q}$.

Assume now that $\zeta > \sqrt{\mathfrak{q}}$ and let q such that $\sqrt{q} \in (\sqrt{\mathfrak{q}}, \zeta)$. We show that this implies $\mathbb{E}[|\log S_T|^p]$ is finite for all $p \in (\mathfrak{q}, q)$. Indeed, in this case, mimicking the previous argument, but recalling here that ζ represents a lim inf, we have that $\sqrt{q} < \zeta$ implies that there exists $x_q < 0$ such that for $x < x_q$, $\mathfrak{d}(x, \mathbf{I}(x)) > \sqrt{2q \log |x|}$, and the exact same steps as above yield

$$\mathbf{I}(x) < \sqrt{2|x| + 2q \log |x|} - \sqrt{2q \log |x|},$$

but this time not only along a subsequence only. From (3.7) it follows that there exists $x^* < 0$ such that for any $x < x^*$,

$$\mathbb{P}_{\text{BS}}(x, \mathbf{I}(x)) < e^x |x|^{-q}.$$

Reverting to moneyness units ($K = e^x$), one sees that for $p \in (q, q)$, $z_p := (e^{p-1}\mathbf{1}_{\{p < 1\}} + \mathbf{1}_{\{p \geq 1\}}) \wedge e^{x^* \wedge x_q}$ is strictly smaller than 1 since $p \geq 0$. In the below, we appeal to Lemma 3.7 applied to the convex function $x \mapsto |\log x|^p \mathbf{1}_{\{x \leq z_p\}} + |\log z_p|^p$, and substitute $x = S_T$, $x_0 = S_0 = 1$ before taking expectations. The final inequality holds since $q > p$ and since $\mathbb{E}[|\log S_T|^p \mathbf{1}_{\{S_T \geq z_p\}}]$ is finite due to S_T being integrable by construction.

$$\begin{aligned} \mathbb{E}[|\log S_T|^p] &= \mathbb{E}[|\log S_T|^p \mathbf{1}_{\{S_T < z_p\}}] + |\log(z_p)|^p + \mathbb{E}[|\log S_T|^p \mathbf{1}_{\{S_T \geq z_p\}}] - |\log(z_p)|^p \\ &= \int_0^{z_p} \frac{\mathbf{P}_{\text{BS}}(\log(K), \mathbf{I}(\log(K)))}{K^2} p \{ (p-1)|\log(K)|^{p-2} + |\log(K)|^{p-1} \} dK + \mathbb{E}[|\log S_T|^p \mathbf{1}_{\{S_T \geq z_p\}}] - |p-1|^p \\ &\leq \int_0^{z_p} \frac{K}{|\log(K)|^q} \frac{p}{K^2} \{ (p-1)|\log(K)|^{p-2} + |\log(K)|^{p-1} \} dK + \mathbb{E}[|\log S_T|^p \mathbf{1}_{\{S_T \geq z_p\}}] - |p-1|^p < \infty. \end{aligned}$$

In the case where $q = 0$, there is nothing to prove about the first part of the proof above. The second part follows the same way with sets of the form (q, \cdot) replaced by $[q, \cdot)$, and the theorem follows. \square

3.3. Refinement of the Fukasawa-Gatheral formula. In his volatility Bible [21], Gatheral derived an elegant formula expressing the log contract directly in terms of the implied volatility. This has obvious appeal as traders can plug in their favourite implied volatility smile (parametric or not) and obtain the fair value of a variance swap. Earlier versions of this formula, albeit with more sketchy proofs, were proposed by Matytsin [36] and Chriss and Morokoff [10]. A fully thorough derivation though has only recently been provided by Fukasawa [19] (see also [33] for interesting connections with absence of arbitrage) who not only proved the key ingredient, the decreasing property of the map $k \mapsto \mathfrak{d}(k, \cdot)$, but extended the formula to more general payoff contract. In all these proofs, the main assumption is the existence of moments $\mathbb{E}[S_T^{1+\varepsilon}]$ for some $\varepsilon > 0$. We show hereafter that this additional condition is in fact not required. Following [19], let

$$(3.8) \quad \mathfrak{f}(x) := -\mathfrak{d}(x, \mathbf{I}(x)) = \frac{x}{\mathbf{I}(x)} + \frac{\mathbf{I}(x)}{2},$$

and note that, as proved by Fukasawa [19], the inverse function \mathfrak{f}^\leftarrow is well defined. This yields the following:

Theorem 3.8. *If $\mathbb{E}[|\log(S_T)|]$ is finite (namely $q \geq 1$) then, with ϕ denoting the Gaussian density,*

$$-2\mathbb{E}[\log(S_T)] = \int_{\mathbb{R}} \mathbf{I}(\mathfrak{f}^\leftarrow(z))^2 \phi(z) dz.$$

Remark 3.9. We should note¹ that a version of the Fukasawa-Gatheral formula was also derived by De Marco and Martini [16], who first removed Roger Lee's $p > 0$ condition (in Theorem 2.3), allowing for $p = 0$. In fact, they considered $\mathbb{E}[\Psi(\log(S_T))]$ where the function Ψ is of exponential growth. Unfortunately, this excludes the identity function as in Theorem 3.8. However, borrowing their argument at the end of their Section 4, this can be achieved by considering the limit of $\mathbb{E}[e^{p \log(S_T)}]$ as p tends to zero since the expectation is at least well defined in a strip of the complex plane of the form $\{p \in \mathbb{C} : \Re(p) \in (0, 1)\}$. The proof we provide below is however more direct and tailored for the log contract.

Remark 3.10. By linearity of the conditional expectation operator, both Put and Call option prices are convex functions, and therefore admit both left- and right-derivatives (but may not be differentiable per se). In the following, in order to avoid heavy notations, all derivatives should be understood as right derivatives (or equivalently all derivatives as left derivatives).

¹We are indebted to the referee for pointing this out explicitly.

Proof. Note first that, by [19, Theorem 2.8] the map $x \mapsto \mathfrak{d}(x, \mathbf{I}(x))$ is decreasing. By (2.1) and the Put-Call parity, a Call option with log-moneyness $x = \log(K/F_T)$ is worth

$$\mathbf{C}_{\text{BS}}(x, \sigma) = \Phi[\mathfrak{d}(x, \sigma) + \sigma] - e^x \Phi[\mathfrak{d}(x, \sigma)].$$

By Lemma 3.7, with $c(e^x) := \mathbf{C}_{\text{BS}}(x, \mathbf{I}(x))$ and $p(e^x) := \mathbf{P}_{\text{BS}}(x, \mathbf{I}(x))$, we can write

$$\begin{aligned} \mathfrak{L} &:= \mathbb{E}[-\log(S_T)] = \int_{-\infty}^0 p(e^x) e^{-x} dx + \int_0^{\infty} c(e^x) e^{-x} dx \\ &= [-p(e^x) e^{-x}]_{-\infty}^0 + [-c(e^x) e^{-x}]_0^{\infty} + \int_{-\infty}^0 p'(e^x) dx + \int_0^{\infty} c'(e^x) dx \\ &= \int_{-\infty}^0 p'(e^x) dx + \int_0^{\infty} c'(e^x) dx. \end{aligned}$$

The boundary terms vanish because $c(1) = p(1)$ by Put-Call parity, because $c(\cdot)$ tends to zero for large strikes and by Lemma 3.3 since $p(x) \leq e^x |x|^{-1} \mathbb{E}[|\log S_T|]$ for $x < 0$ implies $\lim_{x \downarrow -\infty} p(e^x) e^{-x} = 0$. Now,

$$p'(e^x) e^x = \frac{d}{dx} \mathbf{P}_{\text{BS}}(x, \mathbf{I}(x)) \quad \text{and} \quad c'(e^x) e^x = \frac{d}{dx} \mathbf{C}_{\text{BS}}(x, \mathbf{I}(x)).$$

Hence, with $\delta(x) := \mathfrak{d}(x, \mathbf{I}(x))$,

$$\begin{aligned} (3.9) \quad p'(e^x) &= \Phi[-\delta(x)] - \phi(\delta(x)) \delta'(x) + e^{-x} \phi(-\delta(x) - \mathbf{I}(x)) [\delta'(x) + \mathbf{I}'(x)] \\ c'(e^x) &= e^{-x} \phi(\delta(x) + \mathbf{I}(x)) [\delta'(x) + \mathbf{I}'(x)] - \Phi[\delta(x)] - \phi(\delta(x)) \delta'(x). \end{aligned}$$

Since the Gaussian density ϕ satisfies $\phi(a+b) = \phi(a-b) e^{-2ab}$ for any $a, b \in \mathbb{R}$, then

$$e^{-x} \phi(\delta(x) + \mathbf{I}(x)) = e^{-x} \phi\left(-\frac{x}{\mathbf{I}(x)} + \frac{\mathbf{I}(x)}{2}\right) = \phi(\delta(x)),$$

and hence the system (3.9) simplifies, by symmetry of ϕ , to

$$p'(e^x) = \Phi[-\delta(x)] + \phi(\delta(x)) \mathbf{I}'(x) \quad \text{and} \quad c'(e^x) = \phi(\delta(x)) \mathbf{I}'(x) - \Phi[\delta(x)].$$

Therefore

$$\begin{aligned} \mathfrak{L} &= \int_{-\infty}^0 \Phi[-\delta(x)] dx - \int_0^{\infty} \Phi[\delta(x)] dx + \int_{\mathbb{R}} \phi(\delta(x)) \mathbf{I}'(x) dx \\ &= [x \Phi[-\delta(x)]]_{-\infty}^0 - [x \Phi[\delta(x)]]_0^{\infty} + \int_{\mathbb{R}} x \phi(\delta(x)) \delta'(x) dx + \int_{\mathbb{R}} \phi(\delta(x)) \mathbf{I}'(x) dx. \end{aligned}$$

For the boundary terms, observe first from the log-moment Formula, Theorem 3.1, that $\mathfrak{q} \geq 1$ implies $\delta(x) \geq \sqrt{2 \log |x|}$ eventually for $x < 0$, and so $\exp\{-\frac{1}{2} \delta^2(x)\} \leq |x|^{-1}$. Combining with the identity (3.6) one sees $\lim_{x \downarrow -\infty} x \Phi[-\delta(x)] = 0$. Now, Lemma [31, Lemma 3.1] (the right-tail analogue of Lemma 2.2), implies the trivial bound $\mathbf{I}(x) \leq \sqrt{2x}$ for $x > 0$ sufficiently large and therefore

$$\delta(x) = -\left(\frac{x}{\mathbf{I}(x)} + \frac{\mathbf{I}(x)}{2}\right) \leq -\frac{x}{\mathbf{I}(x)} \leq -\frac{\sqrt{x}}{2},$$

which diverges to $-\infty$ as x tends to infinity. Therefore, for x large enough,

$$0 \leq \frac{x \phi(-\delta(x))}{-\delta(x)} = \frac{1}{\sqrt{2\pi}} \frac{x \exp\{-\frac{1}{2} \delta^2(x)\}}{-\delta(x)} \leq \frac{1}{\sqrt{2\pi}} \frac{x \exp\{-\frac{1}{4} x\}}{\frac{x}{\mathbf{I}(x)}} = \frac{\mathbf{I}(x) \exp\{-\frac{1}{4} x\}}{\sqrt{2\pi}} \leq \frac{\sqrt{x} \exp\{-\frac{1}{4} x\}}{\sqrt{\pi}},$$

which tends to zero as x tends to infinity. The limit (3.6) thus implies $\lim_{x \uparrow \infty} x\Phi[\delta(x)] = 0$ and therefore

$$\begin{aligned}\mathfrak{L} &= \int_{\mathbb{R}} x\phi(\delta(x))\delta'(x)dx + \int_{\mathbb{R}} \phi(\delta(x))I'(x)dx \\ &= \int_{\mathbb{R}} x\phi(\delta(x))\delta'(x)dx + [I(x)\phi(\delta(x))]_{\mathbb{R}} + \int_{\mathbb{R}} \phi(\delta(x))\delta(x)\delta'(x)I(x)dx \\ &= \int_{\mathbb{R}} \phi(\delta(x))\delta'(x)[x + I(x)\delta(x)]dx = - \int_{\mathbb{R}} \phi(\delta(x))\delta'(x)\frac{I^2(x)}{2}dx,\end{aligned}$$

where the boundary terms cancel as above and by Lemma 2.2, and applying (2.2) for $\delta(x)$. Substituting $z = \delta(x)$, using the symmetry of ϕ , the proposition follows from the limits $\lim_{x \rightarrow \pm\infty} \delta(x) = \mp\infty$. \square

3.4. Pricing formulae for European options. In [19], Fukasawa not only proved a version of Theorem 3.8 (with more restrictive assumptions), but also extended it to options with payoffs of the form $\Psi(\log(S_T))$ for any twice differentiable function Ψ with derivative of at most polynomial growth. More precisely, he derived [19, Theorem 4.4] an integral form for $\mathbb{E}[\Psi(\log(S_T))]$ assuming either that $\mathbb{E}[S_T^{1+p}]$ exists for some $p > 0$ or that $\mathbb{E}[S_T^{-q}]$ exists for some $q > 0$. The former case is not affected by our setup and we instead provide a refinement of the latter case when no such q exists but instead log-moments are available. This in fact extends the scope of Theorem 3.8 above. Recall that the function \mathfrak{f} is defined in (3.8) and let \mathcal{P}_q^\pm denote the set of functions with at most polynomial growth of order $q \geq 0$ at $\pm\infty$, namely:

$$\mathcal{P}_q^- := \{\Psi : \mathbb{R} \rightarrow \mathbb{R} : \text{there exists } \underline{x} < 0, C > 0 \text{ such that } |\Psi(x)| \leq C(1 + |x|^q) \text{ for all } x \leq \underline{x}\},$$

$$\mathcal{P}_q^+ := \{\Psi : \mathbb{R} \rightarrow \mathbb{R} : \text{there exists } \bar{x} > 0, C > 0 \text{ such that } |\Psi(x)| \leq C(1 + |x|^q) \text{ for all } x \geq \bar{x}\},$$

and let $\mathcal{C}_q^{2,+}$ the set of twice differentiable functions with derivatives in \mathcal{P}_q^+ . We simply write \mathcal{P}^+ (resp. \mathcal{P}^- , $\mathcal{C}^{2,+}$) whenever \mathcal{P}_q^+ (resp. \mathcal{P}_q^- , $\mathcal{C}_q^{2,+}$) holds for any value of $q \geq 0$. We further consider the following assumption on the implied volatility:

Assumption 3.11. There exists $\underline{x} < 0$ such that $I'(x) < 0$ for all $x < \underline{x}$.

This assumption, inconsequential in practice, avoids highly degenerate behaviours of the implied volatility in the left tail. While most – if not all – models used in finance satisfy it, it is possible to construct degenerate models violating it. Such an atypical behaviour may also occur when interpolating option prices instead of implied volatilities [30].

Theorem 3.12. For $\mathfrak{q} \in [1, \infty)$ let $\Psi \in \mathcal{P}_q^-$ with $q \in [0, \mathfrak{q})$ such that $\Psi' \in \mathcal{P}^+ \cap \mathcal{P}_{q'}^-$ with $q' \in [0, \mathfrak{q} - \frac{1}{2})$.

- If Ψ is twice differentiable, then

$$\mathbb{E}[\Psi(\log(S_T))] = \int_{\mathbb{R}} \left\{ \Psi(\mathfrak{f}^\leftarrow(z)) - \Psi'(\mathfrak{f}^\leftarrow(z)) \left[\mathfrak{f}^\leftarrow(z) + \frac{I(\mathfrak{f}^\leftarrow(z))^2}{2} \right] \right\} \phi(z)dz + \int_{\mathbb{R}} \Psi''(x)I(x)\phi(\mathfrak{f}(x))dx.$$

- If Ψ is absolutely continuous, then

$$\mathbb{E}[\Psi(\log(S_T))] = \int_{\mathbb{R}} \left\{ \Psi(\mathfrak{f}^\leftarrow(z)) - \Psi'(\mathfrak{f}^\leftarrow(z)) + \Psi'(\mathfrak{h}(z))e^{-\mathfrak{h}(z)} \right\} \phi(z)dz,$$

where \mathfrak{h} is the inverse function of the map $x \mapsto \mathfrak{f}(x) - I(x)$.

Remark 3.13. The essential difference between this theorem and [19, Theorem 4.4] is the condition on the left tail of the stock price, i.e. on the behaviour of the function Ψ at $-\infty$.

Remark 3.14. With $\Psi(x) \equiv x$, then $\Psi \in \mathcal{P}_1^-$ and $\Psi' \in \mathcal{P}_0^-$, yielding Theorem 3.8.

Proof. The proof of this theorem follows that of [19, Theorem 4.4], or indeed that of Theorem 3.8 above. The steps are analogous, but one has to pay special attention to the boundary terms arising from the different integrations by parts involved. In our setting, the two terms that need special care are

$$(3.10) \quad \lim_{x \downarrow -\infty} \Psi'(x)I(x)\phi(f(x)) \quad \text{and} \quad \lim_{x \downarrow -\infty} \Psi(x)|I'(x)|\phi(f(x)),$$

which we need to send to zero for a suitable class of functions Ψ . The decay to zero at $+\infty$ was taken care of in [19, Theorem 4.4] with the assumption $\Psi \in \mathcal{P}^+$ and the additional two regularity assumptions in the bullet points of the theorem.

By Theorem 3.1, \sqrt{q} is the largest value such that for any $\varepsilon > 0$, there exists x_ε for which

$$(3.11) \quad \frac{\mathfrak{d}(x, I(x))}{\sqrt{2 \log |x|}} > \sqrt{q} - \varepsilon =: \sqrt{q_\varepsilon},$$

for all $x \leq x_\varepsilon$. Now, the equation (in σ) $\frac{\mathfrak{d}(x, \sigma)}{\sqrt{2 \log |x|}} = \sqrt{q_\varepsilon}$ admits two roots $\sigma_\pm = -\sqrt{2q_\varepsilon \log(|x|)} \pm \sqrt{2q_\varepsilon \log(|x|) - 2x}$, so that, for $x < x_\varepsilon$, the inequality (3.11) holds if (similarly to Lemma 3.5 in fact)

$$(3.12) \quad I(x) < -\sqrt{2q_\varepsilon \log(|x|)} + \sqrt{2q_\varepsilon \log(|x|) - 2x}.$$

Note that when $q = 0$ and replacing the \liminf by a genuine limit, this reads $I(x) < \sqrt{2|x|}$ for x small enough, which was proved by Lee [31]. This further implies directly that for $x < x_\varepsilon$,

$$(3.13) \quad f(x) < -\sqrt{2q_\varepsilon \log(|x|)}.$$

Therefore for any function $\Psi : (-\infty, x_\varepsilon] \rightarrow \mathbb{R}$,

$$\Psi'(x)I(x)\phi(f(x)) = \frac{\Psi'(x)I(x)}{\sqrt{2\pi}} \exp\left\{-\frac{f(x)^2}{2}\right\} \leq \frac{\Psi'(x)I(x)}{\sqrt{2\pi}} e^{-q_\varepsilon \log(|x|)} = \frac{\Psi'(x)I(x)}{\sqrt{2\pi}} |x|^{-q_\varepsilon}.$$

From the bound (3.12) on $I(x)$, this expression tends to zero as $x \downarrow -\infty$ if and only if $\Psi' \in \mathcal{P}_{q'}^-$ with $q' \in [0, q - \frac{1}{2})$. Clearly when $q \in [0, \frac{1}{2}]$, this cannot tend to zero as $I(x)$ dominates $\phi(f(x))$. This refines the analysis of [19, Lemma 4.2] which assumed the existence of strictly negative moments for the stock price. Now Fukasawa showed [19, Lemma 2.6] that, independently of any moment (or log-moment) assumptions, $f(x)I'(x) < 1$ for all $x \in \mathbb{R}$; combining this with the new upper bound (3.13), we obtain a new version of [19, Theorem 3.6], namely

$$I'(x) > -\frac{1}{\sqrt{2q \log(|x|)}},$$

for x small enough, so that, by Assumption 3.11, $|I'(x)| < (2q \log(|x|))^{-1/2}$ and therefore

$$\Psi(x)|I'(x)|\phi(f(x)) = \frac{\Psi(x)|I'(x)|}{\sqrt{2\pi}} \exp\left\{-\frac{f(x)^2}{2}\right\} \leq \frac{\Psi(x)|I'(x)|}{\sqrt{2\pi}} e^{-q_\varepsilon \log(|x|)} = \frac{\Psi(x)}{2\sqrt{\pi q}} \frac{|x|^{-q_\varepsilon}}{\sqrt{\log(|x|)}}$$

converges to zero as $x \downarrow -\infty$ as soon as $\Psi \in \mathcal{P}_q^-$ with $q \in [0, q)$. This therefore implies that the two limits (3.10) are equal to zero if and only if $\Psi \in \mathcal{P}_q^-$ for $q \in [0, q)$. All the other statements in [19, Lemma 4.3] remain identical, and therefore the proof of Theorem 4.4 follows analogously, the boundary terms cancelling out under our new assumptions, thus proving the first bullet point in the theorem. A close look at the proof of the second bullet point in [19, Theorem 4.4] shows that only the second limit in (3.10) needs to tend to zero, which, as just discussed, is true as soon as $\Psi \in \mathcal{P}_q^-$, and the theorem follows. \square

4. EXAMPLES

Practically there is no direct market observable for \mathbf{q} (whenever it exists), just as there is none for inverse-power moments. However in a market trading a discretely sampled variance swap our results provide a bound for the left-wing without making further assumptions (restrictions) on the underlying distribution of the stock price. There are two main avenues for leveraging the results in practice. The first is to leverage the bound to help derive a parameterised implied volatility smile. The second is to employ a pricing model in which $\mathbf{q} < \infty$ is an input, for which an initial 'guess' of this value may result from a regression of implied volatility against log-moneyness. We first provide examples of models where only finitely many log-moments are admissible.

4.1. Exponential Lévy models. In exponential Lévy models the stock-price process is modelled by

$$(4.1) \quad S_t = S_0 \exp(L_t),$$

where $(L_t)_{t \in [0, T]}$ is a real-valued Lévy process [39, Chapter 3], namely a càdlàg stochastically continuous process with independent and identically distributed increments starting from $L_0 = 0$. For any $t > 0$, the characteristic function of the random variable L_t satisfies

$$\log \mathbb{E} [e^{iuL_t}] = \psi(u)t,$$

for all $u \in \mathbb{R}$, where the characteristic exponent ψ admits the Lévy-Khintchine representation

$$\psi(u) = -\frac{\xi u^2}{2} + i\gamma u + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x| \leq 1\}}) \nu(dx),$$

with $\xi \geq 0$, $\gamma \in \mathbb{R}$ and ν a measure on \mathbb{R} satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$. Sato [39, Theorem 25.3] proved that for any submultiplicative, locally bounded function g , the expectation $\mathbb{E}[g(S_T)]$ is finite if and only if $\int_{\mathbb{R}} g(x) \nu(dx)$ is finite. In light of Theorem 3.1, we thus consider the function $g(x) \equiv (\log(|x|))^q$ with $q \geq 0$.

4.1.1. Finite moment log stable process. The Finite Moment Log Stable (FMLS) model was introduced by Carr and Wu [9] to capture the negative skew observed on S&P options. There the driving Lévy process L in (4.1) is α -stable with tail index $\alpha \in (1, 2)$ and skew parameter $\beta = -1$, so that [39, Chapter 3], for any $T > 0$,

- $\mathbb{E}[|S_T|^p]$ is finite for all $p \geq 0$;
- the support of L_T is the whole real line;
- $\mathbb{E}[|\log S_T|^q]$ is finite for all $q \in (0, \alpha)$ and is infinite if $q \geq \alpha$.

Theorem 3.1 thus applies with $\mathbf{q} = \alpha \in (1, 2)$ and $\mathbb{E}[|\log(S_T)|^2]$ is infinite. While the model may capture the fat left tail and thin right tail of the stock price, it is too extreme if a discrete variance swap is traded. A more detailed analysis of the left tail of the smile for that model, with higher-order terms, has been derived in [25, Theorem 11.8]. For European option pricing, Theorem 3.12 then applies since $\mathbf{q} = \alpha \geq 1$.

4.1.2. Finite moment log mixture model. In (4.1) let $L := X - Y$ for two independent processes X and Y with

- $\mathbf{q}_X := \sup\{q \geq 0 : \mathbb{E}[|X_1|^q] < \infty\} > 0$ and $\mathbb{E}[e^{\mathbf{p}_X X_1}]$ is finite for some $\mathbf{p}_X \geq 1$;
- $\mathbf{q}_Y := \sup\{q \geq 0 : \mathbb{E}[|Y_1|^q] < \infty\} \in (0, \mathbf{q}_X)$ and $\mathbb{E}[e^{-\mathbf{p}_Y Y_1}]$ for some $\mathbf{p}_Y \in [1, \mathbf{p}_X)$,

so that X and Y respectively influence the right and left tails in the distribution. Before identifying some candidates for the process X and Y , we note:

Lemma 4.1. $\mathbb{E}[e^{\mathbf{p}_Y L_1}]$ is finite and $\mathbf{q}_L := \sup\{q \geq 0 : \mathbb{E}[|L_1|^q] < \infty\} = \mathbf{q}_Y$.

Proof. The first statement follows by independence of X and Y , so that the moment generating function of L is simply the product of those of X and Y . Now, it is clear that $\mathbb{E}|L_1|^q$ is finite for $q < \mathfrak{q}_Y$. For $q > \mathfrak{q}_Y$, observe

$$|Y_1|^q \leq \left| (|Y_1| - |X_1|)^+ + |X_1| \right|^q < 2^q \left(\left\{ (|Y_1| - |X_1|)^+ \right\}^q + |X_1|^q \right)$$

and $(|Y_1| - |X_1|)^+ \leq ||Y_1| - |X_1|| \leq |Y_1 - X_1|$, where this last inequality is due to the reverse triangular inequality. This implies the assertion about \mathfrak{q}_L . \square

Choices for X abound, as any process with finite moments and finite exponential moments of all orders will do, in particular the Brownian motion, the generalised Inverse Gaussian process, the generalised Hyperbolic process [1], the CGMY process [5]. For Y , the choices are scarcer, but the inverse Gaussian process is a valid one, whereby Y is a pure-jump Lévy process with density at time 1 equal to

$$f_{\text{IG}}(y; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\beta/y}, \quad \text{for } y > 0,$$

where $\alpha, \beta > 0$ are the shape and scale parameters and $\Gamma(\cdot)$ is the Gamma function. Jørgensen [29] showed that

$$\mathbb{E}[Y^r] = \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)} \beta^r, \quad \text{if } r < \alpha, \text{ and infinite otherwise.}$$

The reciprocal Gamma distribution is a special case of the Generalised Inverse Gaussian (GIG) distribution and hence is infinitely divisible [1]. With this specification, the log-returns have exploding negative moments beyond order $\mathfrak{q}_L = \alpha$ (possibly larger than 2) and positive moments of arbitrary order depending on X .

4.2. SSVI parameterisation. In [20], Gatheral and Jacquier extended the original Stochastic Volatility Inspired (SVI) parameterisation proposed by Gatheral [20] to a full (strike, maturity) surface. Denoting again x the log-moneyness, they considered the parameterisation

$$I_{\text{SSVI}}(x)^2 = \frac{\theta}{2} \left(1 + \rho \varphi(\theta)x + \sqrt{(\varphi(\theta)x + \rho)^2 + 1 - \rho^2} \right),$$

for all $x \in \mathbb{R}$ with $\rho \in [-1, 1]$, and showed that, with some explicit conditions on θ and $\varphi(\cdot)$, the resulting surface was fully free of arbitrage. Extensions of this framework can be found in [26] and detailed calibration methodology in [27, 35, 34, 37]. Expanding it as x tends to $-\infty$ yields

$$I_{\text{SSVI}}(x) = \sqrt{\frac{\theta(1-\rho)\varphi(\theta)}{2}} \sqrt{|x|} + \frac{c}{\sqrt{|x|}} + \mathcal{O}(|x|^{-3/2}),$$

where the constant c depends explicitly on θ and $\varphi(\theta)$, but we omit the details for clarity. Clearly, this expansion does not fit with Corollary 3.2. Consider instead the ‘SVI-inspired’ formulation

$$\tilde{I}_{\text{SSVI}}(x)^2 := I_{\text{SSVI}}(x)^2 - \frac{\theta}{2} \sqrt{2\mathfrak{q}|x| \log(|x|)},$$

and set $\rho = 0$ and $\varphi(\cdot) = 1/\sqrt{2}$ and $\theta = 4\sqrt{2}$, so that, as x tends to $-\infty$, we obtain

$$\tilde{I}_{\text{SSVI}}(x) = \sqrt{2|x|} - \sqrt{2\mathfrak{q} \log(|x|)} + \frac{\mathfrak{q} \log(|x|)}{\sqrt{2|x|}} + \mathcal{O}(|x|^{-1/2}),$$

which corresponds precisely to the expansion in Corollary 3.2 up to the higher-order error term. We leave the detailed analysis of this modified SSVI volatility surface to future research but simply point out here that, for practical purposes, one should be able to incorporate the optimal moment \mathfrak{q} into a parametric form of volatility surface, immediately usable by practitioners.

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