Robust approximations for pricing Asian options and volatility swaps under stochastic volatility

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Abstract

We show that if the discounted Stock price process is a continuous martingale, then there is a simple relationship linking the variance of the terminal Stock price and the variance of its arithmetic average. We use this to establish a model-independent upper bound for the price of a continuously sampled fixed-strike Arithmetic Asian call option, in the presence of non-zero time-dependent interest rates (Theorem 1.2). We also propose a model-independent lognormal moment-matching procedure for approximating the price of an Asian call, and we show how to apply these approximations under the Black-Scholes and Heston models (subsection 1.3). We then apply a similar analysis to a time-dependent Heston stochastic volatility model, and we show to construct a time-dependent mean reversion and volatility-of-variance function, so as to be consistent with the observed variance swap curve and a pre-specified term structure for the variance of the integrated variance (Theorem 2.1). We characterize the small-time asymptotics of the first and second moments of the integrated variance (Proposition 2.2), and derive an approximation for the price of a volatility swap under the time-dependent Heston model (Eq (50)), using the Brockhaus-Long approximation[BL00]. We also outline a bootstrapping procedure for calibrating a piecewise-linear mean reversion level and volatility-of-volatility function (subsection 2.3.2).

Introduction

Under the Black-Scholes model, there is no closed-form solution for the price of an Arithmetic Asian option. Geman&Yor[GY93] proposed a methodology using time horizon Laplace transforms and properties of time-changed Bessel processes, but their approach only works when the risk-neutral drift is not less than half the squared volatility. This restriction was lifted by Carr&Schroder[CS00] using complex analytic techniques. Donati-Martin,Ghomasni&Yor[DGY01] use time inversion applied to the Brownian exponential functional to show that it has the same law as a certain diffusion process. Levy[Levy92] approximated the price of an Asian option by fitting a lognormal distribution to the Arithmetic Average. Posner&Milevsky[PM98] extended Levy's approach to allow for the exact fitting of the third and fourth moments to a Johnson distribution. In another article, Milevsky&Posner [1998] show that, under certain conditions, when the amount of fixings and

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the time to maturity approach infinity, the law of the arithmetic average approaches an inverse gamma distribution. Rogers&Shi[RS95] derive a sharp lower bound using a general inequality for conditional expectations of a call option payoff, and conditioning on the arithmetic average of the driving Brownian motion; the distribution of the log Stock price conditioned on this random variable is known and normally distributed. Thompson[T99] derives an upper bound which sharpens the upper bound which appears in Rogers&Shi[RS95].

In this article, we take a non-parametric approach. We assume only that the discounted Stock price process is a continuous martingale, and that more than four moments of the terminal Stock price exist. Under these assumptions, we show how to extract the first and second moments of the arithmetic average Stock price, without any further model specification, in terms of the variance of the terminal Stock prices (Theorem 1.1), which we can extract from the observed prices of European options. Using this information, we then establish a robust upper bound for the price of an Asian call, by pricing any quadratic payoff on the average Stock price which dominates the Asian call payoff, and then choosing the cheapest possible payoff function amongst all such quadratics (Theorem 1.2). We show that Theorems 1.1 and 1.2 can be applied to the Black-Scholes and Heston models, and in both cases we present closed-form formulae for the variance of the arithmetic average Stock price. We also propose a lognormal moment-matching approximation (subsection 1.4), which is similar in spirit to the methodology in Levy[Levy92], but here we are using the moments that are implied by European option prices rather than a parametric Black-Scholes model.

In section 3 we propose an extension of the well known Heston stochastic volatility model, with a time-dependent mean reversion level $\theta(t)$, and time-dependent volatility-of-variance $\sigma(t)$. This model has also been discussed in Buehler[Buehler06]. Proceeding along similar lines to Dufresne[Duf01] and Gatheral[Gath07], we show how construct $\theta(t)$ which is consistent with an observed or pre-specified variance swap curve, and a $\sigma(t)$ which is consistent with a pre-specified term structure for the variance of the integrated variance. We discuss how to combine this methodology with the Carr-Lee[CL08] correlation-neutral formula for the second moment of the integrated variance, expressed in terms the price of a European style contract on the underlying Stock. We also derive Taylor series expansions for these quantities in the maturity variable and propose a bootstrapping algorithm for calibrating a piecewise linear $\theta(t)$ and $\sigma(t)$. Finally, we derive a small-time estimate for the volatility swap rate (Eq (50)) by combining these Taylor series with the Brockhaus-Long[BL00] approximation.

1 Robust bounds and approximations for Asian options

1.1 Computing the variance of the average Stock price from the variance of the Stock price

Throughout this section, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\{\mathcal{F}_t\}$, which satisfies the usual conditions of definition 1.2.25 in Karatzas&Shreve[KS91], and we assume that $\mathbb{P}$ is a risk-neutral measure.

Theorem 1.1 *Consider the following stochastic volatility model for a Stock price process S*

$$\begin{aligned} S_t &= e^{\int_0^t r(u)du} \tilde{S}_t \,, \\ d\tilde{S}_t &= \tilde{S}_t \sigma_t dW_t \end{aligned} \qquad (1)$$

under $\mathbb{P}$*, where* W *is a standard Brownian motion. We assume that* $\tilde{S}_t$ *is a non-negative, continuous martingale and* σ_t *is a progressively measurable non-negative volatility process.* $r(t)$ *is a time-dependent interest rate with* $0 \leq r(t) < r_{\max} < \infty$ *for all* t*, and we write* $B_t = e^{\int_0^t r(u)du}$*. Then we have the following relationship between the second moments of* S_t *and the second moment of* $\int_0^t S_u du$

$$\mathbb{E}((\int_0^t S_u du)^2) = 2\int_0^t B_s \int_0^s B_u^{-1}\mathbb{E}(S_u^2)du\, ds. \tag{2}$$

Proof. Let $I_t = \int_0^t S_u du$. I_t is a continuous semimartingale, so we can apply Itô's lemma and take the expectation to obtain

$$\mathbb{E}(I_t^2) = 2\,\mathbb{E}(\int_0^t I_u S_u du)\,. \tag{3}$$

By Fubini's theorem and taking conditional expectations, we can re-write this expression as

$$\begin{aligned}\mathbb{E}(I_t^2) &= 2\int_0^t \mathbb{E}(I_u S_u)du \\ &= 2\int_0^t \int_0^u \mathbb{E}(S_u S_v)dvdu \\ &= 2\int_0^t \int_0^u \mathbb{E}(S_v \mathbb{E}_v S_u)dvdu \\ &= 2\int_0^t B_u \int_0^u B_v^{-1}\mathbb{E}(S_v^2)dvdu\,. \end{aligned} \tag{4}$$

■

1.2 A model-independent upper bound for the price of an Asian call option

Theorem 1.2 *Under the assumptions of Theorem 1.1, we have the following upper bound for the price* P *of a* K*-strike continuously-sampled arithmetic Asian call option which pays* $(I_t - K)^+$ *at time* t

$$P = B_t^{-1}\mathbb{E}(I_t - K)^+ \;\leq\; \frac{B_t^{-1}}{4(K - L^*)}\;\left[\,\mathbb{E}(I_t^2 - 2L^*\mathbb{E}(I_t) + L^{*2}\,\right]\,, \tag{5}$$

where

$$L^* = K - [\,\mathrm{Var}(I_t) + (K - \mathbb{E}(I_t))^2\,]^{\frac{1}{2}}\,, \tag{6}$$

and we can use Theorem 1.1 to write the value of $\mathbb{E}(I_t^2)$ *in terms of* $\mathbb{E}(S_u^2)$ *for* $u \in [0, t]$.

Remark 1.1 *If we can observe European option prices at all strikes and all maturities in* $[0, t]$*, then we can back out* $\mathbb{E}(S_u^2)$ *for all* $u \in [0, t]$ *using the well known Breeden-Litzenberger[BL78] result, and then use this information to compute the Variance of* $\int_0^t S_u du$ *that is implied by the market. We can then plug this value for the Variance into the upper bound in Theorem 1.2.*

Proof. Theorem 1.1 implies that we can compute $\mathbb{E}(\int_0^t S_u du)$ and $\mathbb{E}(\int_0^t S_u du)^2$ if we know the Variance of S_u for all $u \in [0,t]$. Thus we can price *any* quadratic payoff $\mathrm{Q}(I_t)$ on $I_t = \int_0^t S_u du$. Consider the following choice of $Q(.)$

$$\mathrm{Q}(I) = \frac{1}{4(K-L)}\,(I-L)^2\,. \tag{7}$$

with $0 \leq L < K$. $Q(I)$ is always greater than or equal to $(I-K)^+$, and touches $(I-K)^+$ at $I = L$, and $I = U = 2K - L > K$, so we have:

$$\begin{aligned}\mathbb{E}(I_t - K)^+ &\leq \mathbb{E}(\mathrm{Q}(I_t)) \\ &= \frac{1}{4(K-L)}[\,\mathbb{E}(\,I_t^2) - 2L\mathbb{E}(I_t) + L^2\,]\,.\end{aligned} \tag{8}$$

We now wish to find the value of L that minimizes the value of $\mathbb{E}(\mathrm{Q}(I_t))$. Differentiating with respect to L, and setting the derivative to zero we obtain

$$\begin{aligned}\frac{\partial}{\partial L}\,\mathbb{E}(\mathrm{Q}(I_t)) &= \frac{2L - 2(I_t)}{4(K-L)} + \frac{\mathbb{E}(\,I_t^2) - 2L\mathbb{E}(I_t) + L^2}{4(K-L)^2} \\ &= \frac{2(L - \mathbb{E}(I_t))\,(K-L)\, + \mathbb{E}(I_t^2 - 2LI_t + L^2)}{4(K-L)^2} \\ &= 0.\end{aligned} \tag{9}$$

Finally, solving the quadratic in L in the numerator, we see that the relevant root, and thus the optimal value of L, is given by

$$L^* = K - [\,\mathrm{Var}(I_t) + (K - \mathbb{E}(I_t))^2\,]^{\frac{1}{2}} \quad = \quad K - [\mathbb{E}(I_t - K)^2]^{\frac{1}{2}}\,. \tag{10}$$

■

Remark 1.2 *As we would expect by drawing the graph of* Q, L^* *is lower, and its corresponding* U^* *is higher, when the variance of* I_t *is higher, and when the Asian call option is further out-of-the-money (see graph below).*

1.3 Examples

1.3.1 The Black Scholes model

For the Black Scholes model

$$dS_t = rS_t dt + \sigma S_t dW_t\,, \tag{11}$$

the discounted Stock price $\tilde{S}_t = e^{-rt}S_t$ is a martingale, and $\mathbb{E}(S_t^p) < \infty$ for all $p > 1$. Thus we are within the jurisdiction of Theorem 1.1, and the formula in Eq (2) reduces to the following expression

$$\begin{aligned}\mathbb{E}(\int_0^t S_u du)^2 &= \frac{2S_0^2}{t^2}\Big(\frac{e^{(2r+\sigma^2)t}}{(r+\sigma^2)(2r+\sigma^2)} + \frac{1}{r}\big(\frac{1}{2r+\sigma^2} - \frac{e^{rt}}{r+\sigma^2}\big)\Big) \quad \text{for} \quad r > 0\,, \\ &= \frac{2S_0^2}{t^2}\big(\frac{e^{\sigma^2 t} - 1 - \sigma^2 t}{\sigma^4}\big) \qquad \text{for} \quad r = 0\,,\end{aligned} \tag{12}$$

which also appears in Milevsky&Posner[MP98].

1.3.2 The Heston model

Consider the well known Heston model defined by the following stochastic differential equations

$$\begin{aligned} dS_t &= rS_t dt + S_t\sqrt{v_t}\,dW_t\,, && (13)\\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}\,dB_t\,, && (14)\end{aligned}$$

with $d\langle W,B\rangle_t = \rho dt,\ \kappa,\theta,\sigma,v_0 > 0, |\rho| < 1$. The discounted Stock price process $\tilde{S}_t = e^{-rt}S_t$ is a Martingale (see Proposition 2.5 in Andersen&Piterbarg[AP07]). We can compute $\mathbb{E}(S_t^2)$ in closed-form using the analytic continuation of the well known analytic exponential affine characteristic function for the log Stock price (see Heston[Hest93] for a PDE proof using Riccati equations, or Hurd&Kuznetsov[HK08] for a probabilistic treatment). From Eq (2), we find that

$$\begin{aligned}\mathbb{E}((\int_0^t S_u du)^2) &= 2\int_0^t e^{rs}\int_0^s e^{-ru}\mathbb{E}(S_u^2)duds\\ &= S_0^2\int_0^t e^{rs}\int_0^s e^{-ru}\frac{e^{\frac{(\kappa-2\rho\sigma)\kappa\theta u}{\sigma^2}}e^{\frac{2v_0}{\kappa-2\rho\sigma+\gamma\coth(\frac{1}{2}\gamma u)}}}{(\cosh(\frac{1}{2}\gamma u)+\frac{\kappa-2\rho\sigma}{\gamma}\sinh(\frac{1}{2}\gamma u))^{\frac{2\kappa\theta}{\sigma^2}}}duds\\ &= S_0^2(t^2+\frac{1}{3}(r+v_0)t^3+\frac{1}{12}(r^2+rv_0+\kappa(\theta-v_0)+v_0^2+2v_0\sigma\rho)t^4+\mathcal{O}(t^5))\,,\end{aligned} \quad (15)$$

where $\gamma = \sqrt{(\kappa-2\rho\sigma)^2-2\sigma^2}$. For $v_0 = \theta$ and $r = \rho = 0$, we have

$$\mathbb{E}((\int_0^t S_u du)^2) = S_0^2(t^2+\frac{1}{3}v_0t^3+\frac{1}{12}v_0^2t^4+\frac{1}{60}(v_0^3+v_0\sigma^2)t^5+\mathcal{O}(t^6)). \quad (16)$$

We see that the volatility-of-variance term σ only appears on its own at $\mathcal{O}(t^5)$.

1.4 Approximating the price of an Asian option with lognormal Moment-matching

We can proceed along similar lines to Levy[Levy92] and Posner&Milevsky[PM98] by matching the first and second moments of $\int_0^t S_u du$ to a lognormal distribution. We can then approximate the price of an Asian call option as if $X = \log\int_0^t S_u du \sim N(\mu t, \sigma^2 t)$, by solving the following two equations

$$\begin{aligned}\mathbb{E}(e^X) &= e^{(\mu+\frac{1}{2}\sigma^2)t}\,,\\ \mathbb{E}(e^{2X}) &= e^{2(\mu+\sigma^2)t}.\end{aligned} \quad (17)$$

for μ and σ, and then using the Black-Scholes formula to approximate Asian call prices. Recall Remark 1.1.

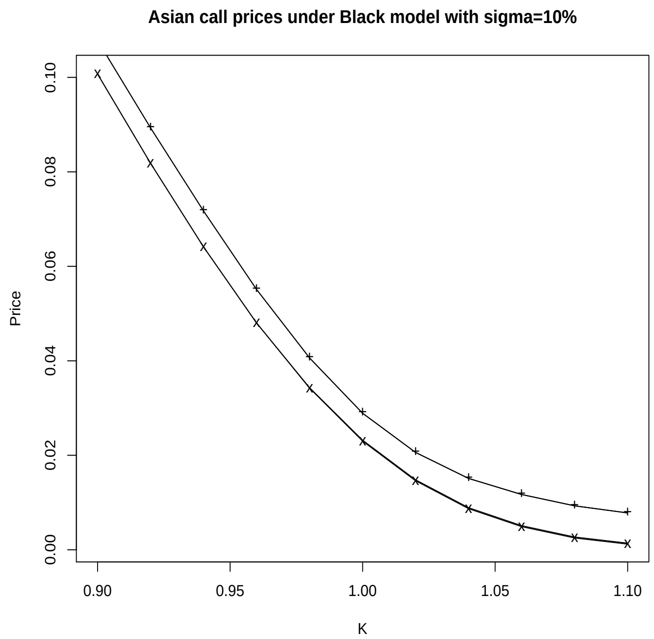

Figure 1: *Here we have tabulated and plotted the prices of ten Arithmetic Asian call options under the Black Scholes model* $dS_t = rS_t dt + \sigma S_t dW_t$, *with* $r = 0$, $\sigma = .10$ *for* $T = 1$ *year maturity, and compared with the Moment-matching approximation and the upper bound in Eq (5). The Monte Carlo prices and the Moment Matched prices are virtually indistinguishable, and we see that the upper bound becomes less useful for Asian options which are heavily out-of-the-money. The true standard deviation of the arithmetic average Stock price is 5.78% (using Eq (12)), which is approximately* $.10/\sqrt{3} = 5.77\%$. *We used an Euler Monte Carlo scheme with 100 time steps and 1,000,000 simulations.*

1.5 Numerical results

K	Monte Carlo	L^*	U^*	Upper bound	Moment-Matching
0.90	0.1007	0.7845	1.0155	0.1078	0.1007
0.92	0.0818	0.8213	1.0187	0.0893	0.0818
0.94	0.0641	0.8597	1.0233	0.0717	0.0641
0.96	0.0480	0.8897	1.0303	0.0551	0.0480
0.98	0.0343	0.9188	1.0412	0.0406	0.0342
1.00	0.0232	0.9422	1.0578	0.0289	0.0230
1.02	0.0148	0.9588	1.0812	0.0206	0.0146
1.04	0.0089	0.9697	1.1103	0.0151	0.0087
1.06	0.0051	0.9767	1.1433	0.0117	0.0049
1.08	0.0027	0.9813	1.1787	0.0093	0.0025
1.10	0.0014	0.9845	1.2155	0.0078	0.0012

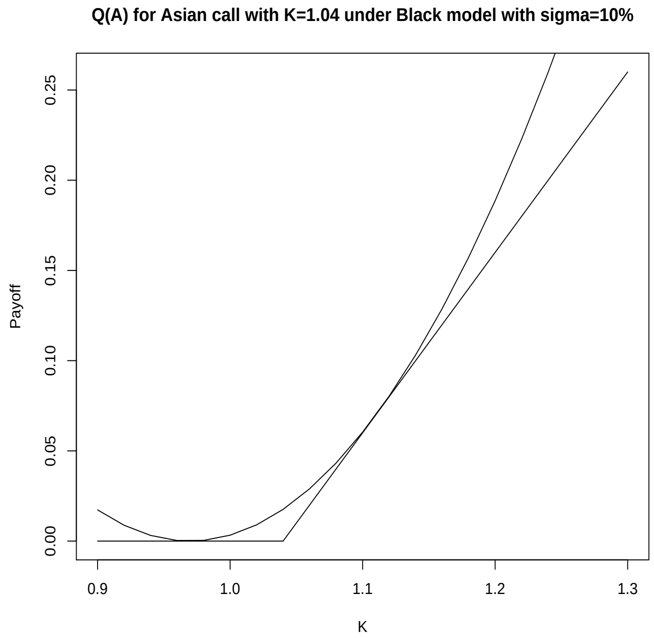

Figure 2: *Here we have plotted the optimal quadratic payoff function for the parameters given in the title of the plot.*

2 Calibrating a time-dependent Heston model and pricing volatility swaps

2.1 Introduction

For a Stock price process S which is a positive continuous martingale under the pricing measure, we can write

$$S_t = S_0\mathcal{E}(X) = S_0 e^{X_t - \frac{1}{2}\langle X\rangle_t}\,, \tag{18}$$

where

$$X_t = \int_0^t \sqrt{v_s}\, dW_s \tag{19}$$

for some Brownian motion W and a predictable square integrable volatility process $\sqrt{v_t}$ (see Buehler[Buehler06]). For the process S, the fair price of a variance swap (at time zero) is $\mathbb{E}_0(\int_0^t v_s ds)$. The variance swap can be replicated with log contracts on S_t, which can in turn be replicated with a static portfolio of European call and put options of the same maturity t (see Neuberger[Neu92] and Appendix A of Buehler[Buehler06] for details).

Brockhaus&Long[BL00] used a second order truncated Taylor series expansion to derive the following approximation

$$\mathbb{E}(\sqrt{\langle X\rangle_t}) \approx \sqrt{\mathbb{E}(\langle X\rangle_t)} - \frac{\mathrm{Var}(\langle X\rangle_t)}{8\mathbb{E}(\langle X\rangle_t)^{\frac{3}{2}}}\,. \tag{20}$$

We define the initial variance swap curve $\textsc{Var}_0(t)$ and the volatility swap curve $\textsc{Vol}_0(t)$ as follows

$$\begin{aligned} \textsc{Var}_0(t) &= \sqrt{\mathbb{E}(\langle X\rangle_t)}\,, \\ \textsc{Vol}_0(t) &= \mathbb{E}(\sqrt{\langle X\rangle_t}) \end{aligned} \tag{21}$$

($\textsc{Vol}(t) \le \textsc{Var}(t)$ by Jensen's inequality). We have made no assumptions about the correlation structure between W and v. Under a general stochastic volatility framework where the correlation ρ between the two Brownian motions is small, Carr&Lee[CL08] derived the following correlation-neutral approximation

$$\mathbb{E}(\langle X\rangle_t^2) \approx \mathbb{E}(4X_t^2 + 16X_t + 8X_t e^{X_t} - 24e^{X_t} + 24) \tag{22}$$

for the second moment of the integrated variance, in terms of the price of a European style contract paying $4X_t^2 + 16X_t + 8X_t e^{X_t} - 24e^{X_t} + 24$ (see section 7.2 in [CL08]). When the correlation is zero, this approximation becomes an equality, and in this special case Carr&Lee[CL08] also derived the following bounds

$$\frac{2\sqrt{\pi}}{S_0}\,\mathbb{E}(S_t - S_0)^+ \le \mathrm{IV}_0(t) \le \textsc{Vol}_0(t) \le \textsc{Var}_0(t) = -2\mathbb{E}(\log(\frac{S_t}{S_0}))\,, \tag{23}$$

where $\mathrm{IV}_0(t)$ is the *unannualized* At-the-Money implied volatility for a European call option of maturity t, and the following approximation for the volatility swap rate

$$\textsc{Vol}_0(\mathrm{t}) \approx \mathrm{IV}_0(t)\,(1 + \frac{\textsc{Var}_0(t)^2 - \mathrm{IV}_0^2(t)}{8 + 2\,\mathrm{IV}_0^2(t)})\,. \tag{24}$$

2.2 Calibrating a time-dependent Heston model

Using a similar analysis to Theorem 1.1, we now construct an extension of the standard Heston stochastic volatility model, with time-dependent mean reversion level and volatility-of-variance, so as to be consistent with an observed or pre-specified variance swap curve, and a pre-specified term structure for the second moment of the integrated variance.

Theorem 2.1 *Consider the following extension of the stochastic volatility model introduced by Heston[Hest93], defined by the following stochastic differential equations*

$$\begin{aligned} dS_t &= rS_t dt + S_t\sqrt{v_t}\, dW_t\,, &(25)\\ dv_t &= \kappa(\theta(t) - v_t)dt + \sigma(t)\sqrt{v_t}\, dB_t\,, &(26)\end{aligned}$$

with $d\langle W, B\rangle_t = \rho dt$, $|\rho| < 1$, $\kappa > 0, v_0 > 0, \theta(t) > 0, \sigma(t) > 0$. *Then we have the following forward equations for* $I_t = \int_0^t v_s ds$

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{E}(I_t^2) &= 2\mathbb{E}(I_t v_t)\,, &(27)\\ \frac{\partial}{\partial t}\mathbb{E}(I_t v_t) &= \mathbb{E}(I_t(\kappa(\theta(t) - v_t)) + \mathbb{E}(v_t^2)\,, &(28)\\ \frac{\partial}{\partial t}\mathbb{E}(v_t^2) &= 2\mathbb{E}(v_t\kappa(\theta(t) - v_t)) + \sigma^2(t)\mathbb{E}(v_t)\,. &(29)\end{aligned}$$

Moreover, we can make this model consistent with a pre-specified variance swap curve $\mathbb{E}(\int_0^t v_s ds)$, *and a pre-specified term structure for the second moment of the integrated variance* $\mathbb{E}((\int_0^t v_s ds)^2)$ *by inverting these equations and choosing* v_0, $\theta(t)$ *and* $\sigma(t)$ *as follows*

$$\begin{aligned} v_0 &= \frac{\partial}{\partial t}\mathbb{E}(\int_0^t v_s ds)|_{t=0}\,, &(30)\\ \theta(t) &= \frac{\frac{\partial}{\partial t}\mathbb{E}(v_t)}{\kappa} + \mathbb{E}(v_t)\,, &(31)\\ \sigma^2(t) &= \frac{\frac{\partial}{\partial t}\mathbb{E}(v_t^2) - 2\kappa\mathbb{E}(v_t(\theta(t) - v_t))}{\mathbb{E}(v_t)}\,, &(32)\end{aligned}$$

if $0 < \theta_{\min} \leq \theta(.) \leq \theta_{\max} < \infty, 0 < \sigma_{\min} \leq \sigma(.) \leq \sigma_{\max} < \infty$, *where*

$$\begin{aligned} \mathbb{E}(v_t^2) &= \frac{\partial}{\partial t}\mathbb{E}(I_t v_t) - \mathbb{E}(I_t(\kappa(\theta(t) - v_t))\,,\\ \mathbb{E}(I_t v_t) &= \frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}(I_t^2)\,. &(33)\end{aligned}$$

Proof. The v_t process satisfies the Yamada-Watanabe condition (see page 291, Proposition 2.13 in Karatzas&Shreve[KS91]), so it admits a unique strong solution (adapted to the natural filtration of (B_t)). Integrating Eq 26, we have that

$$v_t = v_0 + \int_0^t \kappa(\theta(s) - v_s)ds + \sigma(s)\sqrt{v_s}\, dB_s. \qquad (34)$$

From appendix A, we know that $\mathbb{E}(v_s) < \infty$ for all $s \in [0,t]$. Thus $\mathbb{E}(\int_0^t \sigma^2(s) v_s ds) < \infty$, so the stochastic integral $\int_0^t \sigma(s)\sqrt{v_s}\, dB_s$ has zero Expectation. Using Fubini's theorem and the fundamental theorem of calculus, we obtain the forward equation

$$\frac{\partial}{\partial t}\mathbb{E}(v_t) = \kappa(\theta(t) - \mathbb{E}(v_t))\,. \tag{35}$$

We can then re-arrange this equation to recover Eq 31 (note that $\sigma(t)$ does not affect $\mathbb{E}(v_t)$).

We now proceed along similar lines to Dufresne[Duf01]. Recall that $I_t = \int_0^t v_s ds$. By Itô's lemma, we have

$$d(I_t^2) = 2I_t v_t dt\,. \tag{36}$$

Integrating and taking expectations, we obtain

$$\begin{aligned}
\mathbb{E}(I_t^2) = 2\,\mathbb{E}(\int_0^t I_s v_s ds) &= 2\int_0^t \mathbb{E}(I_s v_s) ds \\
&= 2\int_0^t \int_0^s \mathbb{E}(v_s v_u) du ds \\
&\leq 2\int_0^t \int_0^s \mathbb{E}(v_s^2)^{\frac{1}{2}}\mathbb{E}(v_u^2)^{\frac{1}{2}} du ds \\
&< \infty
\end{aligned} \tag{37}$$

using Fubini's theorem and the Schwarz inequality. By the fundamental theorem of calculus, we see that

$$\frac{\partial}{\partial t}\mathbb{E}(I_t^2) = 2\mathbb{E}(I_t v_t). \tag{38}$$

We may repeat this procedure to compute $\mathbb{E}(I_t v_t)$:

$$I_t v_t = \int_0^t (I_s dv_s + v_s dI_s) = \int_0^t I_s(\kappa(\theta(s) - v_s)ds + \sigma(s)\sqrt{v_s}\, dB_s) + v_s^2 ds\,. \tag{39}$$

By the Schwarz inequality, we have that

$$\mathbb{E}(I_t^2 v_t) \leq \mathbb{E}(I_t^4)^{\frac{1}{2}}\mathbb{E}(v_t^2)^{\frac{1}{2}}\,, \tag{40}$$

and (by Jensen's inequality), we know that

$$\frac{1}{t^4}\mathbb{E}(I_t^4) = \mathbb{E}((\frac{1}{t}\int_0^t v_s ds)^4) \;\leq\; \mathbb{E}(\frac{1}{t}\int_0^t v_s^4 ds) = \frac{1}{t}\int_0^t \mathbb{E}(v_s^4) ds\;, \tag{41}$$

and $\mathbb{E}(v_s^4) < \infty$ for $s \in [0,t]$ (see the Appendix). Thus $\mathbb{E}(I_t^4) < \infty$, $\mathbb{E}(v_t^2) < \infty$, $\mathbb{E}(I_t^2 v_t) < \infty$ and

$$\mathbb{E}(\int_0^t \sigma^2(s) I_s^2 v_s ds) < \infty\,, \tag{42}$$

so we have sufficient integrability for the stochastic integral $\int_0^t \sigma(s) I_s \sqrt{v}_s dB_s$ in Eq 39 to have zero expectation. Thus we have the following forward equation for $\mathbb{E}(I_t v_t)$

$$\frac{\partial}{\partial t}\mathbb{E}(I_t v_t) = \mathbb{E}(I_t(\kappa(\theta(t) - v_t)) \,+\, \mathbb{E}(v_t^2). \tag{43}$$

Repeating the procedure again for v_t^2, we find that

$$v_t^2 = v_0^2 + \int_0^t 2v_s(\kappa(\theta(s) - v_s)ds + \sigma(s)\sqrt{v_s}\, dB_s) + \int_0^t \sigma^2(s) v_s \, ds\,. \tag{44}$$

From the Appendix we have that $\mathbb{E}(v_t^3) < \infty$, so the Expectation of the stochastic integral in Eq 44 is also zero, and

$$\frac{\partial}{\partial t}\mathbb{E}(v_t^2) = 2\mathbb{E}(v_t\kappa(\theta(t) - v_t)) \,+\, \sigma^2(t)\mathbb{E}(v_t)\,. \tag{45}$$

Re-arranging this equation, we arrive at Eq (32). ■

2.3 Numerical implementation

Much like Dupire's[Dup94] forward equation for call options under a local volatility model, the equations in Theorem 2.1 look nice on paper, but are difficult to implement in practice because we have to compute higher order derivatives using finite derivatives with noisy/incomplete data. The following subsections address this problem by deriving Taylor series expansions for the expectation and the variance of the integrated variance.

2.3.1 Small-time asymptotic behaviour

Proposition 2.2 *We have the following Taylor series expansions for the expectation and variance of* $\frac{1}{t}\int_0^t v_s ds$ *around* $t = 0$

$$\mathbb{E}(\frac{1}{t}\int_0^t v_s ds) \;=\; v_0 - \frac{1}{2}\kappa(v_0 - \theta(0))t + \frac{\kappa}{6}(\theta'(0) - \kappa(\theta(0) - v_0))t^2 + \mathcal{O}(t^3)\,, \tag{46}$$

$$\mathrm{Var}(\frac{1}{t}\int_0^t v_s ds) \;=\; \frac{1}{3}v_0\sigma(0)^2 t + (-\frac{1}{3}\kappa v_0\sigma(0)^2 + \frac{1}{12}\kappa\theta(0)\sigma(0)^2 + \frac{1}{6}v_0\sigma(0)\sigma'(0))\, t^2 + \mathcal{O}(t^3). \tag{47}$$

Proof. For the first expression, we solve the forward equation Eq 35 for $\mathbb{E}(v_t)$ and integrate to obtain $\mathbb{E}(I_t) = \mathbb{E}(\int_0^t v_s)$, and then expand as a Taylor series. For the second expression, we first solve the forward equation Eq 45 for $\mathbb{E}(v_t^2)$, and expand it as a Taylor series. We then let $\mathbb{E}(I_t v_t) = a_1 t + a_2 t^2 + a_3 t^3 + O(t^4)$, and substitute this expression into Eq 43 and equate coefficients. Finally, we integrate using Eq 3. Note that $\mathrm{Var}(\int_0^t v_s ds) = 0$ if $\sigma(t) \equiv 0$. ■

Remark 2.1 *For small-time asymptotics for call options and implied volatility under the Heston model, see Forde&Jacquier[FJ09].*

2.3.2 A Bootstrapping procedure for piecewise linear $\theta(t)$ and $\sigma(t)$

Proposition 2.3 *Combining Eqs (27), (28) and (29), we have the following Taylor series expansions for the expectation and variance of* I_t *around* $s > 0$ *for* $t > s$

$$\mathbb{E}(I_t) = \mathbb{E}(I_s) + \mathbb{E}(v_s)(t-s) + \frac{\kappa}{2}\left[\theta(s) - \mathbb{E}(v_s)\right](t-s)^2 + \frac{\kappa}{6}\left[\kappa\mathbb{E}(v_s) + \theta'(s) - \kappa\theta(s)\right](t-s)^3 + \mathcal{O}((t-s)^4)\,, \tag{48}$$

and

$$
\begin{aligned}
& \mathrm{Var}(I_t) \\
= \;& \mathrm{Var}(I_s) + 2\left[\mathbb{E}(I_s v_s) - \mathbb{E}(v_s)\mathbb{E}(I_s)\right](t-s) \\
+ \;& \left\{\mathbb{E}(v_s^2) - \mathbb{E}^2(v_s) - \kappa\left[\mathbb{E}(I_s v_s) - \mathbb{E}(I_s)\mathbb{E}(v_s)\right]\right\}(t-s)^2 \\
+ \;& \left\{\frac{1}{3}\kappa^2\left[\mathbb{E}(I_s v_s) - \mathbb{E}(I_s)\mathbb{E}(v_s)\right] + \kappa\left[\mathbb{E}(v_s)^2 - \mathbb{E}(v_s^2)\right] + \frac{\sigma^2(s)}{3}\mathbb{E}(v_s)\right\}(t-s)^3 \\
+ \;& \frac{1}{12}\left\{\kappa^3\left[\mathbb{E}(I_s)\mathbb{E}(v_s) - \mathbb{E}(I_s v_s)\right] + 2\sigma(s)\sigma'(s)\mathbb{E}(v_s) + \kappa\sigma^2(s)\left[\theta(s) - 4\mathbb{E}(v_s)\right] + 7\kappa^2(\mathbb{E}(v_s^2) - \mathbb{E}^2(v_s))\right\}(t-s)^4 \\
+ \;& \mathcal{O}((t-s)^5)\,.
\end{aligned} \tag{49}
$$

Proof. See Appendix B. ■

Remark 2.2 *As a consistency check, we can show that Eq (49) reduces to Eq (47) when $s = 0$, see the end of Appendix B.*

In practice, we can extract the variance swap curve from log contract prices, and we could back out the term structure of the variance of the integrated variance using the Carr-Lee approximation in Eq (22), or we can choose to supply this exogenously. The following procedure outlines how one can use Propositions 2.2 and 2.3 to calibrate a time-dependent Heston model with piecewise linear $\theta(t)$ and $\sigma(t)$:

- Choose a plausible value for κ.
- Fit the variance swap rate curve $\frac{1}{t}\mathbb{E}(\int_0^t v_s ds)$ from $t_0 = 0$ to $t_1 > 0$ with a quadratic, and the $\mathrm{Var}(\frac{1}{t}\int_0^t v_s ds)$ curve with a quadratic, whose leading order term is linear. Using Proposition 2.2, we can then back out $v_0, \theta(0)$ and $\theta'(0)$ from the coefficients of the first quadratic, and $\sigma(0)$ and $\sigma'(0)$ from the coefficients of the second quadratic.
- Using the values for $\theta(0)$, $\theta'(0)$, $\sigma(0)$ and $\sigma'(0)$, construct affine functions for $\theta(t)$ and $\sigma(t)$ over $[0, t_1]$.
- We then solve Eqs (27), (28) and (29) analytically over $[0, t_1]$, so as to compute $\mathbb{E}(v_{t_1}^2), \mathbb{E}(I_{t_1} v_{t_1})$, $\mathbb{E}(I_{t_1}^2)$, and hence $\mathrm{Var}(I_{t_1}^2)$.
- Ignore the $\mathcal{O}((t-s)^4)$ term in Eq (48) and the $\mathcal{O}(t-s)^5)$ term in Eq (49). Then, from pre-specified values for $\mathbb{E}(I_{t_2})$ and $\mathrm{Var}(I_{t_2})$ with $t_2 > t_1$, back out $\theta'(t_1)$ and $\sigma'(t_1)$ using Eqs (48) and (49) respectively with $s = t$ and $t = t_2$. We then use these derivatives to construct affine functions for $\theta(t)$ and $\sigma(t)$ over $[t_1, t_2]$, so that $\theta(t)$ and $\sigma(t)$ are piecewise linear (and continuous) over $[0, t_2]$.
- Repeat this procedure.

2.3.3 Small-time approximation for the volatility swap rate

Combining Proposition 2.2 with the Brockhaus-Long approximation in Eq 20, we obtain the following approximation for the volatility swap rate

$$
\begin{aligned}
\mathbb{E}((\frac{1}{t}\int_0^t v_s ds)^{\frac{1}{2}}) \quad &\approx \quad \sqrt{v_0} + \frac{1}{\sqrt{v_0}}[\frac{1}{4}\kappa(\theta(0)-v_0) - \frac{1}{24}\sigma(0)^2]\,t \\
&+ \quad [\frac{\sigma(0)^2}{32v_0^{\frac{3}{2}}}\kappa(\theta(0)-v_0) - \frac{1}{8v_0^{\frac{3}{2}}}(-\frac{1}{3}v_0\kappa\sigma(0)^2 + \frac{1}{12}\kappa\theta(0)\sigma(0)^2 + \frac{1}{6}v_0\sigma(0)\sigma'(0)) \\
&+ \quad v_0^{\frac{1}{2}}(\frac{1}{12v_0}(\kappa\theta'(0) - \kappa^2\theta(0) + \kappa^2 v_0) - \frac{1}{576v_0^2}\kappa(\theta(0)-v_0)^2)]\,t^2. \qquad (50)
\end{aligned}
$$

From Eq (47), we see that this truncated Taylor series approximation will work better for $\sigma^2(0)t \ll 1$.

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A The time-dependent Heston process as a time-changed Bessel process

Consider the following time-dependent extension of a standard Bessel squared process, which solves the following SDE

$$d(R_t^2) = 2R_t dB_t + \delta(t)dt\,, \tag{51}$$

with $\delta(t)$ time-dependent. Now consider

$$v_t = e^{-\kappa t} R^2_{\int_0^t \frac{1}{4}\sigma^2(s)e^{\kappa s}ds}. \tag{52}$$

Then

$$dv_t = -\kappa v_t + \frac{1}{4}\delta(t)\sigma^2(t)dt + \sigma(t)\sqrt{v_t}dB_t. \tag{53}$$

If we set

$$\delta(t) = \frac{4\kappa\theta(t)}{\sigma^2(t)}\,, \tag{54}$$

then v_t satisfies the same SDE as the time-dependent Heston model

$$dv_t = \kappa(\theta(t) - v_t)dt + \sigma(t)\sqrt{v_t}dB_t. \tag{55}$$

for which we have strong uniqueness. Using the comparison thereom (Proposition 2.18 on page 293 in Karatzas&Shreve[KS91]), we have that

$$\mathbb{P}(R_t \leq R_t^{(1)}, 0 \leq t \leq \infty) = 1, \tag{56}$$

where $R^{(1)}$ is a standard Bessel squared process $BESQ^{\delta_{max}}$, with dimension $\delta_{max} = \max_{0\leq t\leq T}\delta(u)$. The transition density for the $BESQ^{\delta_{max}}$ semigroup is well known (see e.g. Revuz&Yor[RY91]), and R_t^1 has finite moments of all order $0 \leq t < \infty$. Thus we conclude that

$$\mathbb{E}(R_t^8) < \infty\,, \tag{57}$$

and

$$\mathbb{E}(v_t^4) < \infty. \tag{58}$$

B Proof of Proposition 2.3

Let $t \geq 0$. Solving the ODE in Eq (35), and expanding as a Taylor series around $t = s$ and integrating gives

$$\begin{aligned}
\mathbb{E}(I_t) &= \mathbb{E}(I_s) + \mathbb{E}(v_s)(t-s) + \frac{\kappa}{2}\left[\theta(s) - \mathbb{E}(v_s)\right](t-s)^2 \\
&+ \frac{\kappa}{6}\left[\kappa\mathbb{E}(v_s) + \theta'(s) - \kappa\theta(s)\right](t-s)^3 + \frac{\kappa}{24}\left[\kappa^2(\theta - \mathbb{E}(v_s)) + \theta''(s) - \kappa\theta'(s)\right](t-s)^4 + \mathcal{O}((t-s)^5)\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(I_t)^2 &= \mathbb{E}(I_s)^2 + 2\mathbb{E}(I_s)\mathbb{E}(v_s)(t-s) + \left[\kappa\theta(s)\mathbb{E}(I_s) + \mathbb{E}^2(v_s) - \kappa\mathbb{E}(v_s)\mathbb{E}(I_s)\right](t-s)^2 \\
&+ \kappa\left[(\frac{\kappa\mathbb{E}(I_s)}{3} - \mathbb{E}(v_s))(\mathbb{E}(v_s) - \theta(s)) + \frac{1}{3}\mathbb{E}(I_s)\theta'(s)\right](t-s)^3 + \Psi(s,t)(t-s)^4 + \mathcal{O}((t-s)^5)\,,
\end{aligned}$$

where

$$\begin{aligned}
\Psi(s,t) &= \frac{\kappa}{12}\left\{\kappa^2\mathbb{E}(I_s)\left[\theta(s) - \mathbb{E}(v_s)\right] + \mathbb{E}(I_s)\left[\theta''(s) - \kappa\theta'(s)\right] + 7\kappa\mathbb{E}^2(v_s)\right\} \\
&- \frac{5}{6}\mathbb{E}(v_s)\kappa^2\theta(s) + \frac{1}{3}\mathbb{E}(v_s)\kappa\theta'(s) + \frac{1}{4}\kappa^2\theta(s)^2\,.
\end{aligned} \tag{59}$$

We also have

$$\begin{aligned}
\mathbb{E}(v_t^2) &= \mathbb{E}(v_s^2) + \left[-2\kappa\mathbb{E}(v_s^2) + \mathbb{E}(v_s)\sigma^2(s) + 2\kappa\theta(s)\mathbb{E}(v_s)\right](t-s) \\
&+ \left\{\kappa\theta(s)\left[\kappa\theta(s) + \frac{\sigma^2(s)}{2}\right] + \mathbb{E}(v_s)\left[\sigma(s)\sigma'(s) - \frac{3\kappa}{2}\sigma^2(s) + \kappa\theta'(s) - 3\kappa^2\theta(s)\right] + 2\kappa^2\mathbb{E}(v_s^2)\right\}(t-s)^2 \\
&+ \Upsilon(s,t)(t-s)^3 + \mathcal{O}((t-s)^4)\,,
\end{aligned} \tag{60}$$

where

$$\begin{aligned}\Upsilon(s,t) \quad &= -\tfrac{4\kappa^3}{3}\mathbb{E}(v_s^2) + \tfrac{7}{6}\sigma^2(s)\mathbb{E}(v_s)\kappa^2 - \kappa^3\theta^2(s) + \tfrac{1}{3}\mathbb{E}(v_s)\sigma'^2(s) - \tfrac{4}{3}\kappa\sigma(s)\sigma'(s)\mathbb{E}(v_s)\\ &+\tfrac{7}{3}\kappa^3\theta(s)\mathbb{E}(v_s) - \tfrac{1}{2}\sigma^2(s)\kappa^2\theta(s) + \tfrac{1}{6}\sigma^2(s)\kappa\theta'(s) + \kappa^2\theta(s)\theta'(s) + \tfrac{1}{3}\mathbb{E}(v_s)\kappa\theta''(s)\\ &+\tfrac{1}{3}\mathbb{E}(v_s)\sigma(s)\sigma''(s) + \tfrac{2}{3}\sigma(s)\sigma'(s)\kappa\theta(s) - \tfrac{4}{3}\kappa^2\theta'(s)\mathbb{E}(v_s)\,, \end{aligned} \tag{61}$$

and

$$\begin{aligned}\mathbb{E}(I_t v_t) \quad &= \quad \mathbb{E}(I_s v_s) + \left[\kappa\theta(s)\mathbb{E}(I_s) - \kappa\mathbb{E}(I_s v_s) + \mathbb{E}(v_s^2)\right](t-s)\\ &+ \quad \frac{1}{2}\left[\kappa^2(\mathbb{E}(I_s v_s) - \theta(s)\mathbb{E}(I_s)) + 3\kappa(\mathbb{E}(v_s)\theta(s) - \mathbb{E}(v_s^2)) + \sigma(s)^2\mathbb{E}(v_s) + \kappa\theta'(s)\mathbb{E}(I_s)\right](t-s)^2\\ &+ \quad \Xi(s,t)(t-s)^3 + \mathcal{O}((t-s)^4)\,, \end{aligned} \tag{62}$$

where

$$\begin{aligned}\Xi(s,t) \quad &= \quad \frac{1}{6}\left\{\kappa^3\left[\theta(s)\mathbb{E}(I_s) - \mathbb{E}(I_s v_s)\right] + 4\kappa(\theta'(s) - \sigma^2(s))\mathbb{E}(V_s) + \kappa\left[\theta(s)\sigma^2(s) - \kappa\theta'(s)\mathbb{E}(I_s)\right]\right\}\\ &+ \quad \frac{1}{6}\left\{2\mathbb{E}(v_s)\sigma(s)\sigma'(s) - \kappa^2\theta(s)\left[10\mathbb{E}(v_s) - 3\theta(s)\right] + \kappa\theta''(s)\mathbb{E}(I_s) + 7\kappa^2\mathbb{E}(v_s^2)\right\}\,. \end{aligned} \tag{63}$$

Thus we obtain

$$\begin{aligned}\mathbb{E}(I_t^2) \quad &= \quad 2\int_0^t \mathbb{E}(I_u v_u)du\\ &= \quad 2\int_0^s \mathbb{E}(I_u v_u)du + 2\int_s^t \mathbb{E}(I_u v_u)du\\ &= \quad \mathbb{E}(I_s^2) + 2\mathbb{E}(I_s v_s)(t-s) + \left[\kappa\theta(s)\mathbb{E}(I_s) - \kappa\mathbb{E}(I_s v_s) + \mathbb{E}(v_s^2)\right](t-s)^2\\ &+ \quad \frac{1}{3}\left[\kappa^2(\mathbb{E}(I_s v_s) - \theta(s)\mathbb{E}(I_s)) + 3\kappa(\mathbb{E}(v_s)\theta(s) - \mathbb{E}(v_s^2)) + \sigma^2(s)\mathbb{E}(v_s) + \kappa\theta'(s)\mathbb{E}(I_s)\right](t-s)^3\\ &+ \quad \frac{1}{2}\Xi(s,t)(t-s)^4\\ &+ \quad \mathcal{O}((t-s)^5)\,. \end{aligned} \tag{64}$$

Therefore

$$\begin{aligned}\mathrm{Var}(\int_0^t v_u du) \quad &= \quad \left[\mathbb{E}(I_s^2) - \mathbb{E}^2(I_s)\right] + 2\left[\mathbb{E}(I_s v_s) - \mathbb{E}(v_s)\mathbb{E}(I_s)\right](t-s)\\ &+ \quad \left\{\mathbb{E}(v_s^2) - \mathbb{E}^2(v_s) - \kappa\left[\mathbb{E}(I_s v_s) - \mathbb{E}(I_s)\mathbb{E}(v_s)\right]\right\}(t-s)^2\\ &+ \quad \left\{\frac{1}{3}\kappa^2\left[\mathbb{E}(I_s v_s) - \mathbb{E}(I_s)\mathbb{E}(v_s)\right] + \kappa\left[\mathbb{E}^2(v_s) - \mathbb{E}(v_s^2)\right] + \frac{\sigma^2(s)}{3}\mathbb{E}(v_s)\right\}(t-s)^3\\ &+ \quad \left\{\kappa^3\left[\mathbb{E}(I_s)\mathbb{E}(v_s) - \mathbb{E}(I_s v_s)\right] + 2\sigma(s)\sigma'(s)\mathbb{E}(v_s) + \kappa\sigma^2(s)\left[\theta(s) - 4\mathbb{E}(v_s)\right] + 7\kappa^2(\mathbb{E}(v_s^2) - \mathbb{E}^2(v_s))\right\}\frac{(t-s)^4}{12}\\ &+ \quad \mathcal{O}((t-s)^5)\,. \end{aligned} \tag{65}$$

Now, take $s=0$, then $\mathbb{E}(I_s) = 0$, $\mathbb{E}(v_s) = v_0$ and $\mathbb{E}(I_s v_s) = \mathbb{E}(I_s)\mathbb{E}(v_s) = 0$, so we have

$$\mathrm{Var}(\int_0^t v_u du) = \frac{1}{3}\sigma^2(0)v_0 t^3 + \frac{1}{6}\sigma(0)\sigma'(0)v_0 + \frac{\kappa}{12}\sigma^2(0)\left[\theta(0) - 4v_0\right]t^4 + \mathcal{O}(t^5)\,. \tag{66}$$