The Uncertain Volatility Model*

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July 14, 2008

1 Black-Scholes and realised volatility

What happens when a trader uses the Black-Scholes (BS in the sequel) formula to sell and dynamically hedge a call option, at a given constant volatility, whereas the realised volatility will be what it will be, i.e. certainly not a constant?
It is not difficult to show that the answer is the following: if the realised volatility is lower than the managing volatility, the corresponding Profit and Loss (P&L) will be non negative. Indeed, a simple-yet clever application of Itô's formula shows us that the instantaneous P&L of being short a delta-hedged option reads

$$P\&L_t = \frac{1}{2}\Gamma S_t^2 \left[\sigma_t^2 dt - \left(\frac{dS_t}{S_t}\right)^2\right] \tag{1}$$

Where Γ is the Gamma of the option (the second derivative with respect to the underlying, which is positive for a call option), and σ_t the spot volatility, i.e. the volatility at which the option was sold and $\left(\frac{dS_t}{S_t}\right)^2$ represents the realised variance over the period $[t, t + dt]$. Note that this holds without any assumption on the realised volatility which will certainly turn out to be non constant. This result is fundamental in practice: it allows traders to work with neither exact knowledge of the behaviour of the volatility nor a more complex toolbox than the plain Black-Scholes formula; an upper bound of the realised volatility is enough to grant a profit (conversely, a lower bound for option buyers). This way of handling the realised volatility with the Black-Scholes formula is of historical importance in the option market. El Karoui, Jeanblanc and Shreve have formalized it masterfully in [6].

*Keywords : Volatility Risk, EJS paper, convex options and superstrategies

2 Superhedging strategies and the Uncertain Volatility Model (UVM)

2.1 The UVM framework

Fine. Assume you perform the previous strategy. You are certainly not alone in the market, and you wish you have the lowest possible selling price compatible with your risk aversion. In practice on derivatives desk (this is a big difference with the insurance world where the risk is distributed among a large enough number of buyers) the risk aversion is total, meaning that your managing policy will aim at yielding a non negative P&L whatever the realised path. This approach is what is called the superhedging strategy (or superstrategy) approach to derivative pricing. Of course, the larger the set of the underlying scenarii (or paths) for which you want to have the superhedging property, the higher the initial selling price.

The first set which comes to mind is the set of paths associated with an unknown volatility, say between two boundary values $\sigma_{\min}$ and $\sigma_{\max}$. That is, we look for the cheapest price at which we can sell and manage an option, without any assumption on the volatility except that it lies in the $[\sigma_{\min}, \sigma_{\max}]$ range. This framework is the Uncertain Volatility Model (UVM) introduced by Avellaneda, Levy and Paras in [2].

If you take a call option (or more generally a European option with convex payoff), the BS price at volatility $\sigma_{\max}$ is a good candidate. Indeed it yields a superhedging strategy by result (1). And should the realised volatility be constantly $\sigma_{\max}$, then your P&L will be 0. It is easy to conclude from that the BS $\sigma_{\max}$ price is the UVM selling price for an option with a convex payoff.

Now very often traders use strategies (butterflies, callspreads,...) which are not convex any longer. It is not at all easy to find a superstrategy in that case. Except one: if you hedge at the selling time and does not rebalance your hedge before maturity, the cheapest price associated to such a strategy will be the value at the initial underlying value of the concave envelope of the payoff function. It is easy to see that this value corresponds to the total uncertainty case, or yet to the $[0, \infty]$ case in the UVM model. For a call option it will be the value of the underlying.

2.2 The Black-Scholes Barenblatt equation

There comes into play the seminal work [2] and independently [4]. Going back to (1), we are looking for a model with the property that the managing volatility is $\sigma_{\min}$ when the gamma is non negative, and $\sigma_{\max}$ in the converse situation. Should such a model exist, it will yield the optimal solution to the superhedging problem.

An easy way to approximate the optimal solution is to consider a tree (a trinomial tree for instance) where the dependence upon the volatility lies in the node probabilities and not in the tree grid. In the classical backward pricing scheme one can then choose the managing volatility according to the local convexity (since it is a trinomial tree, each node has three children and so a convexity information) of the immediately forward price. Of course, it is not the convexity of the current price since we are calculating it, but the related error of replacing the current convexity by the forward one will certainly go to zero when the time step goes to zero.

The related continuous-time object is the Black-Scholes partial differential equation (PDE) where the second order term is replaced by the non-linear:

$$\frac{1}{2}S_t^2\left(\sigma_{\max}^2\Gamma^+ - \sigma_{\min}^2\Gamma^-\right)$$

Where as usual x^+ and x^- denotes the posive and negative parts. This PDE has been named Black-Scholes-Barenblatt since it looks like the Barenblatt PDE occuring in porosity theory. More precisely, in case of no arbitrage, assume that the stock price dynamics satisfy $dS_t = S_t\left(rdt + \sigma_t dW_t\right)$, where W_t is a standard Brownian motion, r is the risk-free interest rate. This being valid under the class $\mathcal{P}$ of all the probability measures such that $\sigma_{\min} \le \sigma_t \le \sigma_{\max}$. Let Π_t denote the value of a derivative at time t written on S_t with maturity T and final payoff $\Phi\left(S_T\right)$, then at any time $0 \le t \le T$, we must have

$$W^-\left(t, S_t\right) = \inf_{P\in\mathcal{P}} \mathbb{E}_t^P\left[e^{-r(T-t)}\Phi\left(S_T\right)\right] \le \Pi_t \le \sup_{P\in\mathcal{P}} \mathbb{E}_t^P\left[e^{-r(T-t)}\Phi\left(S_T\right)\right] = W^+\left(t, S_t\right)$$

Where the two bounds satisfy the following non-linear PDE, called the Black-Scholes-Barenblatt equation (which obviously reduces to the classical Black-Scholes one in the case $\sigma_{\min} = \sigma_t = \sigma_{\max}$):

$$\boxed{\frac{\partial W^\pm}{\partial t} + r\left(\frac{\partial W^\pm}{\partial S}S - W^\pm\right) + \frac{1}{2}\Sigma\left(\frac{\partial^2 W^\pm}{\partial S^2}\right)S^2\frac{\partial^2 W^\pm}{\partial S^2} = 0}$$

With terminal condition

$$W^\pm\left(S, T\right) = \Phi\left(S_T\right)$$

Where

$$\Sigma\left(\frac{\partial^2 W^+}{\partial S^2}\right) = \begin{cases} \sigma_{\max}^2 & \text{if } \frac{\partial^2 W^+}{\partial S^2} \ge 0 \\ \sigma_{\min}^2 & \text{if } \frac{\partial^2 W^+}{\partial S^2} < 0 \end{cases} \quad \text{and } \Sigma\left(\frac{\partial^2 W^-}{\partial S^2}\right) = \begin{cases} \sigma_{\max}^2 & \text{if } \frac{\partial^2 W^-}{\partial S^2} \le 0 \\ \sigma_{\min}^2 & \text{if } \frac{\partial^2 W^-}{\partial S^2} > 0 \end{cases}$$

Observe that in case Φ is convex, then the Black-Scholes price at volatility $\sigma_{\max}$ is convex for any time t, so that it solves the Black-Scholes-Barenblatt equation. Conversely, if Φ is concave, so is its Black-Scholes price at volatility $\sigma_{\max}$ for any time t, which yields the unique solution to the Black-Scholes-Barenblatt equation.

2.3 Superstrategies and stochastic control

Note that this PDE is also a classical Hamilton-Jacobi-Bellman equation occuring in stochastic control theory. Indeed a related object of interest is the supremum of the risk neutral prices over all the dynamics of volatility that satisfy the range property:

$$\sup_{P \in \mathcal{P}} \mathbb{E}^P f$$

Where $\mathcal{P}$ is the set of risk-neutral probabilities, each of them corresponding to a volatility process with value at each time in $[\sigma_{\min}, \sigma_{\max}]$. In fact such an object is not that easy to define in the classical probabilistic modelling framework, since two different volatility processes will typically yield mutually singular probability measures on the set of possible paths. A convenient framework is the stochastic control framework. In such a framework, the managing volatility being interpreted as a control, one tries to optimize a given expectation-the risk neutral price in that case. It turns out that stochastic optimal control will yield the optimal superstrategy price.

Nevertheless, the connection between the superstrategy problem and stochastic control is not that obvious, and things have to be written carefully in this respect. Recall that the stochastic control problem is the maximization of an expectation over a set of processes, whereas the superstrategy problem is the almost sure domination of the option payoff at maturity by a hedging strategy.

3 Open questions: towards theoretical approach to model risk

Note that even in the UVM case, there are still plenty of open questions. In fact, a neat formulation of the superhedging problem is not a piece of cake. The issue is avoided in [2], handled partially in [4], and more formally in [7], where the model uncertainty is specified as a set of martingale probabilities on the canonical space, and also in [8]. Once this is done, a natural theoretical problem, given such a "model set", is to find out a formula for the cheapest superhedging price. The supremum of the risk neutral prices over all the probabilities of the set will in general be strictly smaller than the cheapest price, even if they match in the UVM setting. The precise property of the "model set" which makes this equality remains to be clarified. Some partial results in this direction, with progresses towards a general theorem, are available in [5], where the case of path-dependent payoffs in the UVM framework is also solved.

4 UVM in practice

4.1 Lagrangian UVM

In practice, the UVM approach is easy to implement for standard options by using the tree scheme described above, for example. It can be extended in the same way for path-dependent options. Nevertheless, when the price pops up, the usual reaction of the trader or risk officer is that the price is too expensive, especially too expensive to explain the observed market price.

The fact that the price is expensive is a direct consequence of the total aversion approach in the superstrategy formulation, and also of the fact that the price corresponds to the worst-case scenario where the Gamma changes signs exactly when the volatility switches regimes. This will hardly happen for real-even if it could.

To lower the price and fit in the traditional setting where one wants to fit the observed market price of liquid European calls and puts (so called vanillas), Avellaneda, Levy and Paras propose a constrained extension of the UVM model where the price of the complex products of the trader is handled with the UVM framework with the additional constraint of fitting the vanilla prices. By duality, this reduces to computing the UVM price for a portfolio parameterised by a Lagragian multiplier and then minimising the dual value function over the Lagrangian parameter. Mathematically speaking, let us consider an asset S_t and a payoff $\Phi(S_T)$. m European options with payoffs $F_1(S_{T_1}), \ldots, F_m(S_{T_m})$ with maybe different strikes and maturities are available for hedging; let $f_1, \ldots, f_m$ their respective market prices at the time of the valuation $t \leq \min(T, T_1, \ldots, T_m)$. Consider now an agent who buys quantities $\lambda_1, \ldots, \lambda_m$ of each option. His total cost of hedging then reads

$$\Pi(t, S_t, \lambda_1, \ldots, \lambda_m) = \sup_{P \in \mathcal{P}} \left\{ e^{-r(T-t)} \Phi(S_T) - \sum_{i=1}^{m} \lambda_i e^{-r(T_i - t)} F_i(S_{T_i}) \right\} + \sum_{i=1}^{m} \lambda_i f_i$$

Where the supremum is calculated within the UVM framework as presented above, and we must specify a range $\Lambda_i^+ \leq \lambda_i \leq \Lambda_i^-$ ($\Lambda_i^{\pm}$ represent the quantities available on the market). The optimal hedge is then defined as the solution of the problem

$$\Pi^*(t, S_t) = \inf_{\lambda_1, \ldots, \lambda_m} \Pi(t, S_t, \lambda_1, \ldots, \lambda_m)$$

In fact, the first-order conditions read $\frac{\partial \Pi}{\partial \lambda_i} = \sum_{i=1}^{m} f_i - \mathbb{E}^{P^*}\left(e^{-r(T_i - t)} F_i(S_{T_i})\right) = 0$, where P^* realises the sup above. These conditions fit exactly the model to observed market prices. The convexity of $\Pi(t, S_t, \lambda_1, \ldots, \lambda_m)$ wrt λ_i ensures that if a minimum exists, then it is unique.

This approach is very seducing from a theoretical point of view, but it is much harder to implement. The consistency of observed vanilla prices is a crucial step which is rarely met in practice. Even if numerous robust algorithms exist to handle the dual problem, their implementation is quite tricky. In fact this constrained formulation implies a calibration property of the model, and the design of a stable and robust calibration algorithm is one of the greatest challenges in the field of financial derivatives.

4.2 The curse of non-linearity

Another issue for practitioner is the inherent non-linearity of the UVM formulation. Most traditional models like Black-Scholes, Heston, or Lévy-based models are linear models. The fact that that an option price should depend on the whole portfolio of the trader is a no-brainer for risk officers, but this non-linearity is a challenge for the modularity and the flexibility of pricing systems. This is very often a no-go feature in practice.

The complexity of evaluating a portfolio in the UVM framework is real, as studied thorougly by Avellaneda and Buff in [1]. Following [1], let us consider a portfolio with n options with payoffs $f_1, \ldots, f_n$ and maturities $t_1, \ldots, t_n$. The computational problem becomes tricky when the portfolio consists of barrier options. Indeed, this means that, at any time step, the portfolio we are trying to value might be different (in case the stock price has reached the barrier of any option) than the one at the previous time step. Because of the non-linearity, a PDE specific to this portfolio has to be solved in this case. In [1], Avellaneda and Buff addressed this very specific issue : a naive implementation would require solving the $2^n - 1$ non-linear PDE, each representing a subportfolio. They provide an algorithm to build the minimal number N_n of subportfolios (i.e. of non-linear PDE to solve) and show that

- If the initial portfolio consists of barrier (single or double) and vanilla options, then $N_n \leq \frac{n(n+1)}{2}$
- If the initial portfolio only consists of single barrier options (n_u up-and-out and $n_d = n - n_u$ down-and-out), then $N_n = n_d + n_u + n_d n_u$. This assumes that all the barrier are different. If some are identical, then the number of required computations decreases.

Numerically speaking, the finite-difference pricing is done on a lattice matching almost exactly all the barriers. Still in [3], they provide an optimal construction of the lattice to solve the PDEs.

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