# $\mathfrak{X P D E}$ FOR $\mathfrak{X} \in\{\mathrm{BS}, \mathrm{FBS}, \mathrm{P}\}:$ A ROUGH VOLATILITY CONTEXT 

OFELIA BONESINI AND ANTOINE JACQUIER


#### Abstract

Recent mathematical advances in the context of rough volatility have highlighted interesting and intricate connections between path-dependent partial differential equations and backward stochastic partial differential equations. In this note, we make this link precise-in Proposition 2.5 identifying the slightly obscure random field introduced in 4] as a pathwise derivative of the value function.


## 1. Introduction

Since the simple yet far-reaching observation by Gatheral, Jaisson and Rosenbaum [12-and supported by more in-depth evidence since [6] [ that historical volatility of major financial indices and single stocks exhibited a behaviour akin to that of fractional Brownian motion, research has flourished in this new rough volatility field; more advanced models have been proposed [1, 4, 8, 13, new statistical methods have been developed to estimate the Hölder regularity of the volatility process [6, 10 and subtle numerical Monte Carlo schemes-with rates of convergence-have been devised 3, 7, 5, 2, 9, 11,

While classical Markovian stochastic volatility models can be handled by PDE techniques thanks to Feynam-Kać, this approach was not obviously applicable in this new, non-Markovian, rough volatility setting. Viens and Zhang 17 however proved that an Itô formula was valid, opening the gates to PDE-based tools, albeit in an infinite-dimensional setting. Inspired by this intuition, Bayer, Qiu and Yao 44 and Bonesini, Jacquier and Pannier 77 set forth an analysis respectively based on a Backward Stochastic PDE (BSPDE) approach and on a path-dependent PDE (PPDE) formulation, covering both theoretical well-posedness of such equations and tackling their numerical and convergence aspects. We note in passing that other investigations of a numerical scheme (based on neural networks) for such PPDEs were also pushed forward in [14, 16, yet leaving aside the exact study of existence and uniqueness.

In this note, we shed light on a question left hanging loose after reading these last two papers: since both develop a framework for the value function-say the price of a European option-the two obtained representations, proved to be unique, should thus coincide. In Section 2, we first recall these two setups (Section 2.1 for Bayer-Qiu-Yao and Section 2.2 for Bonesini-Jacquier-Pannier), compare them precisely in Section 2.3 and provide the missing identification in Section 2.4. Because these two frameworks are in fact more general than simple rough volatility models, we make it explicit in Section 3 how they take shape in such context, of major interest in mathematical finance.

[^0]
## 2. The identification of the BSPDEs

2.1. Bayer-Qiu-Yao: From $\mathfrak{X}=\mathrm{BS}$ to $\mathfrak{X}=$ FBS. In 4], Bayer, Qiu and Yao investigate the weak solution theory to the following BSPDE [4] Equation (2.1)], in the sense of distributions:

$$
\begin{align*}
& \mathrm{d} \breve{u}_{t}(x)=\psi_{t}(x) \mathrm{d} W_{t}  \tag{2.1}\\
& \quad-\left\{\frac{\breve{V}_{t}}{2} \mathrm{D}^{2} \breve{u}_{t}(x)+\rho \sqrt{\breve{V}_{t}} \mathrm{D} \psi_{t}(x)-\frac{\breve{V}_{t}}{2} \mathrm{D} \breve{u}_{t}(x)+F_{t}\left(\mathrm{e}^{x}, \breve{u}_{t}(x), \bar{\rho} \sqrt{\breve{V}_{t}} \mathrm{D} \breve{u}_{t}(x), \psi_{t}(x)+\rho \sqrt{\breve{V}_{t}} \mathrm{D} \breve{u}_{t}(x)\right)\right\} \mathrm{d} t,
\end{align*}
$$

for $(t, x) \in[0, T) \times \mathbb{R}$, with boundary condition $\breve{u}_{T}(\cdot)=G\left(\mathrm{e}^{\cdot}\right)$ on $\mathbb{R}$, where $\boldsymbol{W}=(W, B)$ is a standard two-dimensional Brownian motion, $V$ a given continuous, non-negative, integrable, process adapted to $\mathcal{F}^{W}$, D denotes the total derivative, and $\bar{\rho}:=\sqrt{1-\rho^{2}}$ with $\rho \in[-1,1]$. Note that the searched solution is the couple ( $\breve{u}, \psi$ ) (as a coupled random field), and not only $\breve{u}$. The functions $F$ and $G$ are assumed to satisfy the following:

## Assumption 2.1.

- The map $G:\left(\Omega \times \mathbb{R}, \mathcal{F}_{T}^{W} \otimes \mathcal{B}(\mathbb{R})\right) \rightarrow(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has at most linear growth;
- The function $F:\left(\Omega \times[0, T] \times \mathbb{R}^{4}, \mathcal{F}^{W} \otimes \mathcal{B}\left(\mathbb{R}^{4}\right)\right) \rightarrow(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfies
- Lipschitzianity in its last three variables almost surely;
- linear growth in the hyperplane ( $\mathbb{R}, 0,0,0$ ) almost surely;
$-\left|F_{t}\left(x, y, z_{1}, z_{2}\right)-F_{t}(x, y, 0,0)\right| \leq L_{0}$ almost surely for some $L_{0}>0$ for all $x, y, z_{1}, z_{2} \in \mathbb{R}$.
To the BSPDE (2.1), Bayer, Qiu and Yao naturally associate the FBSDE, for $0 \leq t \leq s \leq T$,

$$
\left\{\begin{align*}
\breve{X}_{s}^{t, x}= & x-\int_{t}^{s} \frac{\breve{V}_{t}}{2} \mathrm{~d} r+\int_{t}^{s} \rho \sqrt{\breve{V}_{t}} \mathrm{~d} W_{r}+\int_{t}^{s} \bar{\rho} \sqrt{\breve{V}_{t}} \mathrm{~d} B_{r}  \tag{2.2}\\
\breve{Y}_{s}^{t, x}= & G\left(\mathrm{e}^{\breve{X}_{T}^{t, x}}\right)-\int_{s}^{T} \breve{Z}_{r}^{t, x} \cdot \mathrm{~d} \boldsymbol{W}_{r} \\
& +\int_{s}^{T} F_{r}\left(\mathrm{e}^{x}, \breve{u}_{r}\left(\breve{X}_{r}^{t, x}\right), \bar{\rho} \sqrt{\breve{V}_{r}} \mathrm{D} \breve{u}_{r}\left(\breve{X}_{r}^{t, x}\right), \psi_{r}\left(\breve{X}_{r}^{t, x}\right)+\rho \sqrt{\breve{V}_{r}} \mathrm{D} \breve{u}_{r}\left(\breve{X}_{r}^{t, x}\right)\right) \mathrm{d} r
\end{align*}\right.
$$

for a two-dimensional process $\breve{Z}^{t, x}$, the existence and characterisation of which are given as follows:
Theorem 2.2 (Theorem 2.4 in [4). Let Assumption 2.1 hold and assume that ( $\breve{u}, \psi)$ is a weak solution to (2.1) with at most exponential growth for $\breve{u}_{t}$ for each $t$. Then $(\breve{u}, \psi)$ admits a version satisfying

$$
\breve{Y}_{s}^{t, x}=\breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right), \quad \breve{Z}_{s}^{t, x}=\left(\breve{Z}_{1, s}^{t, x}, \breve{Z}_{2, s}^{t, x}\right)=\left(\bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right), \rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)+\psi_{s}\left(\breve{X}_{s}^{t, x}\right)\right)
$$

almost surely for all $s \in[t, T]$, where $\left(\breve{Y}^{t, x}, \breve{\boldsymbol{Z}}^{t, x}\right)$ satisfies the BSDE (2.2).
Let us further stress that, applying [4. Lemma 2.3] to (2.1) and exploiting the dynamics (2.2),

$$
\begin{align*}
\mathrm{d} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)= & F_{s}\left(\mathrm{e}^{x}, \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right), \bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)+\rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)\right) \mathrm{d} s  \tag{2.3}\\
& +\left(\psi_{s}\left(\breve{X}_{s}^{t, x}\right)+\rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)\right) \mathrm{d} W_{s}+\bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right) \mathrm{d} B_{s} .
\end{align*}
$$

2.2. Bonesini-Jacquier-Pannier: The $\mathfrak{X}=\mathrm{P}$ approach. In 7, Bonesini, Jacquier and Pannier consider the system

$$
\left\{\begin{array}{l}
\boldsymbol{X}_{t}=x+\int_{0}^{t} \boldsymbol{b}\left(t, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\int_{0}^{t} \boldsymbol{\sigma}\left(t, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r} \\
\boldsymbol{\Theta}_{s}^{t}=x+\int_{0}^{t} \boldsymbol{b}\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} r+\int_{0}^{t} \boldsymbol{\sigma}\left(s, r, \boldsymbol{X}_{r}\right) \mathrm{d} \boldsymbol{W}_{r}
\end{array}\right.
$$

for $0 \leq t \leq s \leq T$, where $x \in \mathbb{R}^{d}, \mathbf{W}$ is an $m$-dimensional Brownian motion, $\boldsymbol{\sigma}(\cdot)$ takes values in $\mathbb{R}^{d \times m}$ and $\boldsymbol{b}(\cdot)$ in $\mathbb{R}^{d}$. Conditions on $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ do not need to be explictly stated (as argued in [7) as long as the system admits a weak solution such that all moments of $\boldsymbol{X}$ are finite, but would like to emphasise that the process $\boldsymbol{X}$ is clearly not Markovian in general. . For functionals of these processes, [7. Theorem 2.11], based on [17. Theorem 3.17], established the functional Itô formula

$$
\begin{align*}
\mathrm{d} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right)= & \left\{\partial_{t} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right)+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right),\left(\boldsymbol{\sigma}^{t, \boldsymbol{X}}, \boldsymbol{\sigma}^{t, \boldsymbol{X}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right), \boldsymbol{b}^{t, \boldsymbol{X}}\right\rangle\right\} \mathrm{d} t \\
& +\left\langle\partial_{\boldsymbol{\omega}} u\left(t, \boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}\right), \boldsymbol{\sigma}^{t, \boldsymbol{X}}\right\rangle \mathrm{d} \boldsymbol{W}_{t}, \tag{2.4}
\end{align*}
$$

with boundary condition $u(T, \boldsymbol{X})=\Phi(\boldsymbol{X})$, with $\Phi: \mathcal{W} \rightarrow \mathbb{R}^{d^{\prime}}$ for some $d^{\prime} \in \mathbb{N}, \mathcal{W}:=\mathcal{C}^{0}\left([0, T], \mathbb{R}^{d}\right)$ is the space of continuous functions from $[0, T]$ to $\mathbb{R}^{d}$ and $\varphi^{t, \boldsymbol{\omega}}(s):=\varphi\left(s ; t, \boldsymbol{\omega}_{t}\right)$, for $\varphi \in\{b, \sigma\}$ and $\boldsymbol{\omega} \in \mathcal{W}$. The concatenation operator $\otimes$ is borrowed from 17 and should be understood as $\boldsymbol{X} \otimes_{t} \boldsymbol{\Theta}^{t}:=$ $\boldsymbol{X} \mathbb{1}_{[0, t]}+\boldsymbol{\Theta}^{t} \mathbb{1}_{[t, T]}$. We also refer to $[17$ for a precise definition of the inner product $\langle\cdot, \cdot\rangle$. Define the forward-backward stochastic Volterra equation, for $\boldsymbol{\omega} \in \mathcal{W}$ and all $0 \leq t \leq s \leq T$ :

$$
\left\{\begin{array}{l}
\boldsymbol{X}_{s}^{t, \boldsymbol{\omega}}=\boldsymbol{\omega}_{s}+\int_{t}^{s} \boldsymbol{b}\left(s, r, \boldsymbol{X}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r+\int_{t}^{s} \boldsymbol{\sigma}\left(s, r, \boldsymbol{X}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} \boldsymbol{W}_{r}  \tag{2.5}\\
\boldsymbol{Y}_{s}^{t, \boldsymbol{\omega}}=\Phi\left(\boldsymbol{X}^{t, \boldsymbol{\omega}}\right)+\int_{s}^{T} f\left(r, \boldsymbol{X}^{t, \boldsymbol{\omega}}, \boldsymbol{Y}_{r}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r-\int_{s}^{T} \boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}} \cdot \mathrm{~d} \boldsymbol{W}_{r}
\end{array}\right.
$$

with $\boldsymbol{\omega}_{0}=x$. The forward process $\boldsymbol{X}$ lives in $\mathbb{R}^{d}$ and the backward one $\boldsymbol{Y}$ in $\mathbb{R}^{d^{\prime}}$. Moreover the function $f:[0, T] \times \mathcal{W} \times \mathbb{R}^{d^{\prime}} \times \mathbb{R}^{d^{\prime} \times d} \rightarrow \mathbb{R}^{d^{\prime}}$ is assumed to be measurable in all variables. Under a suitable set of assumptions, the backward SDE admits a unique square integrable solution [18, Theorem 4.3.1]. In [7, Proposition 2.14], the authors further proved that $u(t, \boldsymbol{\omega}):=\boldsymbol{Y}_{t}^{t, \boldsymbol{\omega}}$ satisfies the semi-linear path-dependent PDE (in the classical sense)

$$
\begin{equation*}
\partial_{t} u(t, \boldsymbol{\omega})+\frac{1}{2}\left\langle\partial_{\boldsymbol{\omega} \boldsymbol{\omega}} u(t, \boldsymbol{\omega}),\left(\boldsymbol{\sigma}^{t, \boldsymbol{\omega}}, \boldsymbol{\sigma}^{t, \boldsymbol{\omega}}\right)\right\rangle+\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \boldsymbol{b}^{t, \boldsymbol{\omega}}\right\rangle+f\left(t, \boldsymbol{\omega}, u(t, \boldsymbol{\omega}),\left\langle\partial_{\boldsymbol{\omega}} u(t, \boldsymbol{\omega}), \boldsymbol{\sigma}^{t, \boldsymbol{\omega}}\right\rangle\right)=0 \tag{2.6}
\end{equation*}
$$

for $t \in[0, T)$, with boundary condition $u(T, \boldsymbol{\omega})=\Phi(\boldsymbol{\omega})$. Now, applying Itô's formula (2.4) and using the PPDE (2.6), we can write, for $0 \leq t \leq s \leq T$,
$\mathrm{d} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)=-f\left(s, u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right),\left\langle\partial_{\boldsymbol{\omega}} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \boldsymbol{\sigma}^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle\right) \mathrm{d} t+\left\langle\partial_{\boldsymbol{\omega}} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \boldsymbol{\sigma}^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle \mathrm{d} \boldsymbol{W}_{s}$,
where $\widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, r}:=\boldsymbol{X}^{t, \boldsymbol{\omega}} \otimes_{r} \widetilde{\boldsymbol{X}}_{r, \cdot}^{t, \boldsymbol{\omega}}$ denotes the concatenation process, with $\widetilde{\boldsymbol{X}}^{t, \boldsymbol{\omega}}$ defined as the (unique, weak) solution to

$$
\widetilde{\boldsymbol{X}}_{r, s}^{t, \boldsymbol{\omega}}=\boldsymbol{\omega}_{s}+\int_{t}^{r} \boldsymbol{b}\left(s, r^{\prime}, \boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}}\right) \mathrm{d} r^{\prime}+\int_{t}^{r} \boldsymbol{\sigma}\left(s, r^{\prime}, \boldsymbol{X}_{r^{\prime}}^{t, \boldsymbol{\omega}}\right) \mathrm{d} \boldsymbol{W}_{r^{\prime}}
$$

for any $0 \leq t \leq r \leq s \leq T$. Combined with the boundary condition and the fact that the BSDE (2.5) has a unique solution, then

$$
\left(\boldsymbol{Y}_{s}^{t, \boldsymbol{\omega}}, \boldsymbol{Z}_{s}^{t, \boldsymbol{\omega}}\right)=\left(u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right),\left\langle\partial_{\boldsymbol{\omega}} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \boldsymbol{\sigma}^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle\right)
$$

is in fact the unique solution to the BSDE (2.5).
2.3. Comparison. We compare in this section the two setups in the most general setting possible, namely that described in 4, which does not assume a specific structure for the process $V$, as opposed to the (rather general yet of specific Volterra form) one used in 7. The key idea is to exploit the fact that both the BSPDE (2.1) and the PPDE (2.6) are linked to an FBSDE possessing a unique solution and to identify the terms. First of all, set the coefficients in $(2.5)$ to be

$$
\boldsymbol{X}=\binom{V}{X}, \quad \boldsymbol{W}=\binom{W}{B}, \quad \boldsymbol{b}(t, r, \boldsymbol{x})=\binom{b(t, r, v)}{-\frac{1}{2} v}, \quad \boldsymbol{\sigma}(t, r, \boldsymbol{x})=\left(\begin{array}{cc}
\sigma(t, r, v) & 0  \tag{2.7}\\
\rho \sqrt{v} & \bar{\rho} \sqrt{v}
\end{array}\right)
$$

with $W$ and $B$ independent one-dimensional Brownian motions and $\boldsymbol{x}=(v, x) \in[0, \infty) \times \mathbb{R}$.

Remark 2.3. This framework is slightly less general than the one in 4, since the latter makes no assumption whatsoever on the form of the dynamics of $V$. Note that the coefficient in position $(1,2)$ in the matrix $\boldsymbol{\sigma}(\cdot)$ is null, reflecting the assumption, as in 4 and all cases of interest in quantitative finance, that the process $V$ is adapted to $\mathcal{F}^{W}$ and not to $\mathcal{F}^{B}$.

In full coordinates, we then have, starting from time $t=0$,

$$
\left\{\begin{aligned}
V_{t} & =v+\int_{0}^{t} b\left(t, r, V_{r}\right) \mathrm{d} r+\int_{0}^{t} \sigma\left(t, r, V_{r}\right) \mathrm{d} W_{r}, \\
X_{t} & =x-\frac{1}{2} \int_{0}^{t} V_{r} \mathrm{~d} r+\rho \int_{0}^{t} \sqrt{V_{r}} \mathrm{~d} W_{r}+\bar{\rho} \int_{0}^{t} \sqrt{V_{r}} \mathrm{~d} B_{r},
\end{aligned}\right.
$$

for some $v>0$, and so, for the forward part of the FBSDE (2.5), we can write, for $0 \leq t \leq s \leq T$,

$$
\left\{\begin{aligned}
V_{s}^{t, \omega} & =\omega_{s}+\int_{t}^{s} b\left(s, r, V_{r}^{t, \omega}\right) \mathrm{d} r+\int_{t}^{s} \sigma\left(s, r, V_{r}^{t, \omega}\right) \mathrm{d} W_{r} \\
X_{s}^{t,(\omega, x)} & =x-\frac{1}{2} \int_{t}^{s} V_{r}^{t, \omega} \mathrm{~d} r+\rho \int_{t}^{s} \sqrt{V_{r}^{t, \omega}} \mathrm{~d} W_{r}+\bar{\rho} \int_{t}^{s} \sqrt{V_{r}^{t, \omega}} \mathrm{~d} B_{r}
\end{aligned}\right.
$$

with $\omega_{0}=v$. Note that the functionals studied in 4 only depend on the terminal value of the second component, namely $\Phi\left(\left(\boldsymbol{x}_{\boldsymbol{t}}\right)_{t \in[0, T]}\right)=G\left(\mathrm{e}^{x_{T}}\right)$. Furthermore, the function $F(\cdot)$ in [4] (that is in 2.1) and (2.2)) determining the dynamics of $\breve{Y}$ does not depend on the whole path of $\breve{X}$ as well, but just on the value of its second component at a specific time, that is

$$
f\left(r,\left(\boldsymbol{x}_{\boldsymbol{t}}\right)_{t \in[0, T]}, y, \boldsymbol{z}\right)=F_{r}\left(\mathrm{e}^{x_{r}}, y, \boldsymbol{z}\right)
$$

so that we can now write the backward part of the FBSDE 2.5

$$
Y_{s}^{t,(\omega, x)}=G\left(\mathrm{e}^{X_{T}^{t,(\omega, x)}}\right)+\int_{s}^{T} F_{r}\left(\mathrm{e}^{X_{r}^{t,(\omega, x)}}, Y_{r}^{t,(\omega, x)}, \boldsymbol{Z}_{r}^{t, \omega, x}\right) \mathrm{d} r-\int_{s}^{T} \boldsymbol{Z}_{r}^{t, \omega, x} \cdot \mathrm{~d} \boldsymbol{W}_{r}
$$

The slight abuse of notation between the path $\boldsymbol{\omega}$ in 2.5 and the (path, point) couple ( $\omega, x$ ) should hopefully not create any confusion.

Remark 2.4. It is clear that this is the same FBSDE as in 4 Equation (2.2)]-see also (2.2) above-and so, by the aforementioned identification of terms, we can deduce interesting identities.

As already noticed, in this case the value function $u$ is a function of $X_{t}$ only and so the PPDE (2.6) reduces to

$$
\begin{aligned}
& \partial_{t} u+\left\langle\partial_{\omega} u, b^{t, \omega_{t}}\right\rangle+\left\langle\partial_{x} u,-\frac{\omega_{t}}{2}\right\rangle+F_{t}\left(\mathrm{e}^{x}, u,\left\langle\partial_{\omega} u, \sigma^{t, \omega_{t}}\right\rangle+\rho \sqrt{\omega_{t}} \partial_{x} u, \bar{\rho} \sqrt{\omega_{t}} \partial_{x} u\right) \\
& +\frac{1}{2}\left\{\left\langle\partial_{\omega \omega} u,\left(\sigma^{t, \omega_{t}}, \sigma^{t, \omega_{t}}\right)\right\rangle+\left\langle\partial_{\omega \omega} u,(0,0)\right\rangle+\left\langle\partial_{x \omega} u,\left(\rho \sqrt{\omega_{t}}, \sigma^{t, \omega_{t}}\right)\right\rangle\right. \\
& \left.+\left\langle\partial_{\omega x} u,\left(\rho \sqrt{\omega_{t}}, 0\right)\right\rangle+\left\langle\partial_{x x} u,\left(\rho \sqrt{\omega_{t}}, \rho \sqrt{\omega_{t}}\right)\right\rangle+\left\langle\partial_{x x} u,\left(\bar{\rho} \sqrt{\omega_{t}}, \bar{\rho} \sqrt{\omega_{t}}\right)\right\rangle\right\}=0,
\end{aligned}
$$

with $u$ evaluated at $(t, \omega, x)$, and with boundary condition $u(T, \omega, x)=G\left(\mathrm{e}^{x}\right)$. Removing the redundant terms simplifies this expression to

$$
\begin{aligned}
& \partial_{t} u+\frac{1}{2}\left\{\left\langle\partial_{\omega \omega} u,\left(\sigma^{t, \omega_{t}}, \sigma^{t, \omega_{t}}\right)\right\rangle+\rho \sqrt{\omega_{t}}\left\langle\partial_{\omega}\left(\partial_{x} u\right), \sigma^{t, \omega_{t}}\right\rangle+\omega_{t} \partial_{x x} u\right\}+\left\langle\partial_{\omega} u, b^{t, \omega_{t}}\right\rangle-\frac{\omega_{t}}{2} \partial_{x} u \\
& +F_{t}\left(\mathrm{e}^{x}, u,\left\langle\partial_{\omega} u, \sigma^{t, \omega_{t}}\right\rangle+\rho \sqrt{\omega_{t}} \partial_{x} u, \bar{\rho} \sqrt{\omega_{t}} \partial_{x} u\right)=0
\end{aligned}
$$

with boundary condition $u(T, \omega, x)=G\left(\mathrm{e}^{x}\right)$. We conclude this section with a clarification of the dependency of $u$ on the paths of $\omega$ and on the point $x$. In matrix form, the dynamics of the state process $\boldsymbol{X}$ reads

$$
\boldsymbol{X}_{s}=\binom{V_{s}}{X_{s}}=\binom{\omega_{0}}{x}+\int_{0}^{s}\binom{b\left(s, r, V_{r}\right)}{-\frac{1}{2} V_{r}} \mathrm{~d} r+\int_{0}^{s}\left(\begin{array}{cc}
\sigma\left(s, r, V_{r}\right) & 0 \\
\rho \sqrt{V_{r}} & \bar{\rho} \sqrt{V_{r}}
\end{array}\right)\binom{\mathrm{d} W_{r}}{\mathrm{~d} B_{r}}
$$

$$
=:\binom{\Theta_{s}^{t}}{\Xi_{s}^{t}}+\int_{t}^{s}\binom{b\left(s, r, V_{r}\right)}{-\frac{1}{2} V_{r}} \mathrm{~d} r+\int_{t}^{s}\left(\begin{array}{cc}
\sigma\left(s, r, V_{r}\right) & 0 \\
\rho \sqrt{V_{r}} & \bar{\rho} \sqrt{V_{r}}
\end{array}\right)\binom{\mathrm{d} W_{r}}{\mathrm{~d} B_{r}} .
$$

By uniqueness of solutions, $\boldsymbol{X}=\boldsymbol{X}^{t, \boldsymbol{X} \otimes_{t} \Theta^{t}}$, or equivalently $V=V^{t, V \otimes_{t} \Theta^{t}}$ and $X=X^{t,\left(V \otimes_{t} \Theta^{t}, X_{t}\right)}$, where we have exploited the fact that the equation for $X$ does not explicitly depend on time, and so

$$
\begin{align*}
Y_{s}^{t, \boldsymbol{X}} & =G\left(\mathrm{e}^{X_{T}^{t,\left(V \otimes_{t} \Theta^{t}, X_{t}\right)}}\right)+\int_{s}^{T} F_{r}\left(\mathrm{e}^{X_{r}^{t,\left(V \otimes_{t} \Theta^{t}, X_{t}\right)}}, Y_{r}^{t, \boldsymbol{X}}, \boldsymbol{Z}_{r}^{t, \boldsymbol{X}}\right) \mathrm{d} r-\int_{s}^{T} \boldsymbol{Z}_{r}^{t, \boldsymbol{X}} \cdot \mathrm{~d} W_{r}  \tag{2.8}\\
& =u\left(s, V \otimes_{t} \Theta^{t}, X_{t}\right)=u\left(s, \Theta^{t}, X_{t}\right)
\end{align*}
$$

where in the last step we have exploited the fact that, being $T$ and $r \in[s, T]$ greater that $t$, the "past" ( $V, X$ ) does not matter and what matter is $\left(\Theta^{t}, \Xi^{t}\right)$ (and in our specific case only $\left(\Theta^{t}, X_{t}\right)$ ). This actually means that we can write $Y_{s}^{t,\left(V \otimes_{t} \Theta^{t}, X_{t}\right)}$ (resp. $\boldsymbol{Z}_{r}^{t,\left(V \otimes_{t} \Theta^{t}, X_{t}\right)}$ ). Let us mention that, with a slight abuse of notation, we keep the notation $u$ for the solution even when omitting the useless dependencies.
2.4. Identification of the terms. The key identification clarifying the slightly obscure random field $\psi$ introduced in 4 reads as follows:

Proposition 2.5. The following identity holds for all $0 \leq t \leq s$ :

$$
\psi_{s}\left(\breve{X}_{s}^{t, x}\right)=\left\langle\partial_{\omega} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle .
$$

Proof. Since the processes $\breve{Y}$ and $Y$ (and hence $\breve{\boldsymbol{Z}}$ and $\boldsymbol{Z}$ ) solve the same equation, namely the backward part of $(2.2)$ (in differential form) and $(2.8)$ (in integral form) respectively, for which existence and uniqueness hold (from Theorem 2.2 and 77 Proposition 2.14] respectively), the identities are

$$
\begin{aligned}
\text { (Bayer-Qiu-Yao [4]) } & \longleftrightarrow \quad \text { (Bonesini-Jacquier-Pannier } \mathbf{7} \text { ) } \\
\breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)=\breve{Y}_{s}^{t, x} & \longleftrightarrow \boldsymbol{Y}_{s}^{t,(\omega, x)}=u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \\
\left(\bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right), \rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)+\psi_{s}\left(\breve{X}_{s}^{t, x}\right)\right)=\breve{Z}_{s}^{t, x} & \longleftrightarrow \boldsymbol{Z}_{s}^{t,(\omega, x)}=\left\langle\partial_{\boldsymbol{\omega}} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \boldsymbol{\sigma}^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle
\end{aligned}
$$

where, using the formulation of $\boldsymbol{\sigma}$ in (2.7), the last term can be expanded as

$$
\left\langle\partial_{\boldsymbol{\omega}} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \boldsymbol{\sigma}^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle=\left(\bar{\rho} \sqrt{\Theta_{s}^{t}} \partial_{x} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \rho \sqrt{\Theta_{s}^{t}} \partial_{x} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)+\left\langle\partial_{\omega} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle\right) .
$$

This then yields two identifications: the first one reads $\bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)=\bar{\rho} \sqrt{\Theta_{s}^{t}} \partial_{x} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)$, which in particular implies $\sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)=\sqrt{\Theta_{s}^{t}} \partial_{x} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)$, while the second is

$$
\begin{aligned}
\rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)+\psi_{s}\left(\breve{X}_{s}^{t, x}\right) & =\rho \sqrt{\Theta_{s}^{t}} \partial_{x} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right)+\left\langle\partial_{\omega} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle \\
& =\rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)+\left\langle\partial_{\omega} u\left(s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}\right), \sigma^{s, \widehat{\boldsymbol{X}}^{t, \boldsymbol{\omega}, s}}\right\rangle
\end{aligned}
$$

and the lemma follows.

## 3. The prototypical rough volatility model case

The case of European option pricing in a prototypical rough volatility model (where the volatility process is driven by a Riemann-Liouville fractional Brownian motion) provides a nice example that may help the reader think about the general case.
3.1. Comparison of the BSDEs associated to the price of a European option. In the FBSDEs approach, $F_{s}(\cdot) \equiv 0$, and so (2.3) simplifies substantially, in the sense of distributions, to

$$
\left\{\begin{align*}
\mathrm{d} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right) & =\left(\psi_{s}\left(\breve{X}_{s}^{t, x}\right)+\rho \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)\right) \mathrm{d} W_{s}+\bar{\rho} \sqrt{\breve{V}_{s}} \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right) \mathrm{d} B_{s}  \tag{3.1}\\
\breve{u}_{T}\left(\breve{X}_{T}^{t, x}\right) & =G\left(\mathrm{e}^{\breve{X}_{T}^{t, x}}\right)
\end{align*}\right.
$$

On the PPDE side, in the rough volatility setting, Equation 2.5 reduces to the couple of onedimensional equations

$$
\left\{\begin{aligned}
X_{s}^{t,(x, \omega)} & =x-\frac{1}{2} \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right)^{2} \mathrm{~d} r+\rho \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right) \mathrm{d} W_{r}+\bar{\rho} \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right) \mathrm{d} B_{r} \\
V_{s}^{t, \omega} & =\omega_{s}+\int_{t}^{\text {s. }} K(s-r) \mathrm{d} W_{r}
\end{aligned}\right.
$$

for a suitable function $\chi$, and where $K(\cdot)$ represents the kernel of the fractional Brownian motion.
Remark 3.1. For this model to satisfy the assumptions in 4, we need to set $\chi\left(t, V_{t}\right)=\sqrt{V_{t}}$.
In this setting, the functional Itô formula applied to $u_{t}=u\left(t, X_{t}, \Theta^{t}\right)=\mathbb{E}\left[G\left(\mathrm{e}^{X_{T}}\right) \mid \mathcal{F}_{t}\right]$ leads to

$$
\begin{align*}
\mathrm{d} u_{t}= & \left\{\partial_{t} u_{t}+\frac{1}{2} \chi\left(t, V_{t}\right)^{2}\left(\partial_{x x} u_{t}-\partial_{x} u_{t}\right)+\rho \chi\left(t, V_{t}\right)\left\langle\partial_{\omega}\left(\partial_{x} u_{t}\right), K^{t}\right\rangle+\frac{1}{2}\left\langle\partial_{\omega \omega} u_{t},\left(K^{t}, K^{t}\right)\right\rangle\right\} \mathrm{d} t \\
& +\left\{\rho \chi\left(t, V_{t}\right) \partial_{x} u_{t}+\left\langle\partial_{\omega} u_{t}, K^{t}\right\rangle\right\} \mathrm{d} W_{t}+\bar{\rho} \chi\left(t, V_{t}\right) \partial_{x} u_{t} \mathrm{~d} B_{t}  \tag{3.2}\\
= & \left\{\rho \chi\left(t, V_{t}\right) \partial_{x} u_{t}+\left\langle\partial_{\omega} u_{t}, K^{t}\right\rangle\right\} \mathrm{d} W_{t}+\bar{\rho} \chi\left(t, V_{t}\right) \partial_{x} u_{t} \mathrm{~d} B_{t},
\end{align*}
$$

where $K^{t}(\cdot)=K(\cdot-t)$ and where in the last line we have exploited the fact that the drift should be null by the martingale property. Thus, rewriting $(3.2)$ in an extended form we have

$$
\left\{\begin{align*}
\mathrm{d} u\left(t, X_{t}, \Theta^{t}\right) & =\left\{\rho \chi\left(t, V_{t}\right) \partial_{x} u\left(t, X_{t}, \Theta^{t}\right)+\left\langle\partial_{\omega} u\left(t, X_{t}, \Theta^{t}\right), K^{t}\right\rangle\right\} \mathrm{d} W_{t}+\bar{\rho} \chi\left(t, V_{t}\right) \partial_{x} u\left(t, X_{t}, \Theta^{t}\right) \mathrm{d} B_{t}  \tag{3.3}\\
u\left(T, X_{T}, \Theta^{T}\right) & =G\left(\mathrm{e}^{X_{T}}\right)
\end{align*}\right.
$$

Now, mimicking the proof of Proposition 2.5, we identify the terms in (3.1) and (3.3) respectively, and we obtain two representations: On the one hand, $\bar{\rho} \sqrt{ } \breve{V}_{t} \mathrm{D} \breve{u}_{t}\left(\breve{X}_{t}^{0, x}\right)=\bar{\rho} \chi\left(t, V_{t}\right) \partial_{x} u\left(t, X_{t}, \Theta^{t}\right)=$ $\bar{\rho} \sqrt{V_{t}} \partial_{x} u\left(t, X_{t}, \Theta^{t}\right)$ and therefore

$$
\mathrm{D} \breve{u}_{t}\left(\breve{X}_{t}^{0, x}\right)=\partial_{x} u\left(t, X_{t}, \Theta^{t}\right)
$$

on the other hand,

$$
\begin{aligned}
\rho \sqrt{\breve{V}_{t}} \mathrm{D} \breve{u}_{t}\left(\breve{X}_{t}^{0, x}\right)+\psi_{t}\left(\breve{X}_{t}^{0, x}\right) & =\rho \chi\left(t, V_{t}\right) \partial_{x} u\left(t, X_{t}, \Theta^{t}\right)+\left\langle\partial_{\omega} u\left(t, X_{t}, \Theta^{t}\right), K^{t}\right\rangle \\
& =\rho \sqrt{\breve{V}_{t}} \mathrm{D} \breve{u}_{t}\left(\breve{X}_{t}^{0, x}\right)+\left\langle\partial_{\omega} u\left(t, X_{t}, \Theta^{t}\right), K^{t}\right\rangle .
\end{aligned}
$$

and hence $\psi_{t}\left(\breve{X}_{t}^{0, x}\right)=\left\langle\partial_{\omega} u\left(t, X_{t}, \Theta^{t}\right), K^{t}\right\rangle$.
Remark 3.2. This first identification highlights the fact that in 4 , the authors have to 'somehow' hide the stochasticity inside $\breve{u}_{t}$ and only emphasises that $\breve{u}_{t}(x)$ is $\mathcal{F}_{t}^{W}$-measurable. In particular, this is what makes it necessary for them to introduce the additional process $\psi_{t}(x)$ which, as we showed, is the pathwise derivative of $u$ itself, namely $\left\langle\partial_{\omega} u\left(t, X_{t}, \Theta^{t}\right), K^{t}\right\rangle$.
3.2. Identification of the terms in the backward part of the FBSDEs. In the rough volatility setting, the FBSDE in (4)-see also 2.2 -reads

$$
\left\{\begin{align*}
\mathrm{d} \breve{Y}_{s}^{t, x} & =-\breve{Z}_{1, s}^{t, x} \mathrm{~d} W_{s}-\breve{Z}_{2, s}^{t, x} \mathrm{~d} B_{s}, & \breve{Y}_{T}^{t, x} & =G\left(\mathrm{e}^{\breve{X}_{T}^{t, x}}\right),  \tag{3.4}\\
\mathrm{d} \breve{X}_{s}^{t, x} & =-\frac{1}{2} \breve{V}_{s} \mathrm{~d} s+\sqrt{\breve{V}_{s}}\left(\rho \mathrm{~d} W_{s}+\bar{\rho} \mathrm{d} B_{s}\right), & \breve{X}_{t}^{t, x} & =x
\end{align*}\right.
$$

Under their set of assumptions this equation has a unique solution and, in particular, Theorem 2.2 gives an explicit representation for such a solution. As highlighted at the beginning of the previous subsection, on the PPDEs' side, in the rough volatility setting, Equation 2.5 reduces to the following couple of one-dimensional equations

$$
\left\{\begin{aligned}
X_{s}^{t,(x, \omega)} & =x-\frac{1}{2} \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right)^{2} \mathrm{~d} r+\rho \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right) \mathrm{d} W_{r}+\bar{\rho} \int_{t}^{s} \chi\left(r, V_{r}^{t, \omega}\right) \mathrm{d} B_{r} \\
V_{s}^{t, \omega} & =\omega_{s}+\int_{t}^{\text {s. }} K(s, r) \mathrm{d} W_{r}
\end{aligned}\right.
$$

while the backward equation (2.5 becomes

$$
\begin{equation*}
\mathrm{d} Y_{r}^{t,(x, \omega)}=-\boldsymbol{Z}_{r}^{t,(x, \omega)} \cdot \mathrm{d} \boldsymbol{W}_{r}, \tag{3.5}
\end{equation*}
$$

with boundary condition $Y_{T}^{t,(x, \omega)}=G\left(\mathrm{e}^{X_{T}^{t,(x, \omega)}}\right)$. In particular, for the process $\boldsymbol{Z}^{t, \boldsymbol{\omega}}$, we have

$$
\begin{aligned}
& \boldsymbol{Z}_{r}^{t, \boldsymbol{\omega}}=\left[\begin{array}{l}
Z_{1, r}^{t, \boldsymbol{\omega}} \\
Z_{2, r}^{t, \boldsymbol{\omega}}
\end{array}\right]=\left\langle\partial_{\boldsymbol{\omega}} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right), \sigma^{\left.r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right\rangle}\right\rangle \\
& =\left\langle\left[\begin{array}{l}
\partial_{x} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right) \\
\partial_{\omega} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right)
\end{array}\right],\left[\begin{array}{cc}
\bar{\rho} \chi\left(r, V_{r}^{t, \omega}\right) & \rho \chi\left(r, V_{r}^{t, \omega}\right) \\
0 & K(r, \cdot)
\end{array}\right]\right\rangle \\
& =\left[\begin{array}{c}
\bar{\rho} \partial_{x} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right) \chi\left(r, V_{r}^{t, \omega}\right) \\
\rho \partial_{x} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right) \chi\left(r, V_{r}^{t, \omega}\right)+\left\langle\partial_{\omega} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right), K^{r}\right\rangle
\end{array}\right],
\end{aligned}
$$

and so

$$
\left\{\begin{array}{l}
Z_{1, r}^{t, \omega}=\bar{\rho} \partial_{x} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right) \chi\left(r, V_{r}^{t, \omega}\right) \\
Z_{2, r}^{t, \omega}=\rho \partial_{x} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right) \chi\left(r, V_{r}^{t, \omega}\right)+\left\langle\partial_{\omega} u\left(r, X_{r}^{t,(x, \omega)}, \Theta^{t,(x, \omega)}\right), K^{r}\right\rangle
\end{array}\right.
$$

In particular, since Equations (3.4) and (3.5) admit unique solutions and are actually the same equation, we can conclude that

$$
\breve{Y}_{s}^{t, x}=Y_{s}^{t,(x, \omega)}, \quad \breve{Z}_{1, s}^{t, x}=Z_{2, s}^{t,(x, \omega)}, \quad \breve{Z}_{2, s}^{t, x}=Z_{2, s}^{t,(x, \omega)}
$$

and therefore
$\breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)=u\left(s, X_{s}^{t,(x, \omega)}\right), \quad \mathrm{D} \breve{u}_{s}\left(\breve{X}_{s}^{t, x}\right)=\partial_{x} u\left(s, X_{s}^{t,(x, \omega)}\right), \quad \psi_{s}\left(\breve{X}_{s}^{t, x}\right)=\left\langle\partial_{\omega} u\left(s, X_{s}^{t,(x, \omega)}\right), K^{s}\right\rangle$.
Remark 3.3. Notice that we already knew (only) the following identity

$$
\breve{Y}_{t}^{t, x}=\breve{u}_{t}\left(\breve{X}_{t}^{t, x}\right)=\mathbb{E}\left[G\left(\mathrm{e}^{\breve{X}_{T}^{t, x}}\right) \mid \mathcal{F}_{t}\right]=\mathbb{E}\left[G\left(\mathrm{e}^{X_{T}}\right) \mid \mathcal{F}_{t}\right]=u\left(t, X_{t}, \Theta^{t}\right)=Y_{t}^{t,(x, \omega)}
$$

## References

[1] E. Abi Jaber, M. Larsson, and S. Pulido, Affine Volterra processes, The Annals of Applied Probability, 29 (2019), pp. 3155-3200.
[2] C. Bayer, P. Friz, and J. Gatheral, Pricing under rough volatility, Quantitative Finance, 16 (2016), pp. 887-904.
[3] C. Bayer, E. J. Hall, and R. Tempone, Weak error rates for option pricing under linear rough volatility, International Journal of Theoretical and Applied Finance, 2250029 (2023).
[4] C. Bayer, J. Qiu, and Y. Yao, Pricing options under rough volatility with backward SPDEs, SIAM Journal on Financial Mathematics, 13 (2022), pp. 179-212.
[5] M. Bennedsen, A. Lunde, and M. S. Pakkanen, Hybrid scheme for Brownian semistationary processes, Finance and Stochastics, 21 (2017), pp. 931-965.
[6] A. E. Bolko, K. Christensen, M. S. Pakkanen, and B. Veliyev, A GMM approach to estimate the roughness of stochastic volatility, Journal of Econometrics, 235 (2023), pp. 745-778.
[7] O. Bonesini, A. Jacquier, and A. Pannier, Rough volatility, path-dependent PDEs and weak rates of convergence, arXiv:2304.03042, (2023).
[8] O. El Euch, M. Fukasawa, and M. Rosenbaum, The microstructural foundations of leverage effect and rough volatility, Finance and Stochastics, 22 (2018), pp. 241-280.
[9] P. K. Friz, W. Salkeld, and T. Wagenhofer, Weak error estimates for rough volatility models, arXiv:2212.01591, (2022).
[10] M. Fukasawa, T. Takabatake, and R. Westphal, Consistent estimation for fractional stochastic volatility model under high-frequency asymptotics, Mathematical Finance, 32 (2022), pp. 1086-1132.
[11] P. Gassiat, Weak error rates of numerical schemes for rough volatility, SIAM Journal on Financial Mathematics, (2023).
[12] J. Gatheral, T. Jaisson, and M. Rosenbaum, Volatility is rough, Quantitative finance, 18 (2018), pp. 933-949.
[13] A. Jacquier, A. Muguruza, and A. Pannier, Rough multifactor volatility for SPX and VIX options, arXiv:2112.14310, (2021).
[14] A. Jacquier and M. Oumgari, Deep curve-dependent pdes for affine rough volatility, SIAM Journal on Financial Mathematics, 14 (2023), pp. 353-382.
[15] S. E. Rømer, Empirical analysis of rough and classical stochastic volatility models to the SPX and VIX markets, Quantitative Finance, 22 (2022), pp. 1805-1838.
[16] Y. F. Saporito and Z. Zhang, Path-dependent deep Galerkin method: A neural network approach to solve pathdependent partial differential equations, SIAM Journal on Financial Mathematics, 12 (2021), pp. 912-940.
[17] F. Viens and J. Zhang, A martingale approach for fractional Brownian motions and related path-dependent PDEs, The Annals of Applied Probability, 29 (2019), pp. 3489-3540.
[18] J. Zhang and J. Zhang, Backward stochastic differential equations, Springer, 2017.


[^0]:    Department of Mathematics, Imperial College London
    Department of Mathematics, Imperial College London, and the Alan Turing Institute
    E-mail addresses: obonesin@ic.ac.uk, a.jacquier@imperial.ac.uk.
    Date: September 20, 2023.
    2020 Mathematics Subject Classification. 60G22, 35K10, 65C20, 91G20, 91G60.
    Key words and phrases. Rough volatility, path-dependent PDEs, weak rates, stochastic Volterra equations.
    OB and AJ are supported by the EPSRC grant EP/T032146/1. The authors are grateful to C. Bayer, A. Pannier and C. Cuchiero for insightful comments.

